CONNECTION PROBLEM FOR PAINLEVÉ TAU FUNCTIONS

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Dedicated to my wife DeJonette and my daughter Tallulah.

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I express my highest gratitude to my advisor Professor Alexander Its. He was very patient with me and always was there to help. He showed me his great passion and immense experience in doing mathematical research. I am very thankful that he agreed to guide me through this graduate school.

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I would not be able to go to the graduate school without my teachers in Saint-Petersburg University, Nikolai Filonov and Vladimir Sukhanov as well without the guidance of my mother Elena.

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ABSTRACT

Prokhorov Andrei Ph.D., Purdue University, August 2019. Connection Problem for Painlevé Tau Functions. Major Professor: Alexander Its.

We derive the differential identities for isomonodromic tau functions, describing their monodromy dependence. For Painlevé equations we obtain them from the relation of tau function to classical action which is a consequence of quasihomogeneity of corresponding Hamiltonians. We use these identities to solve the connection problem for generic solution of Painlevé-III(D8) equation, and homogeneous Painlevé-II equation.

We formulate conjectures on Hamiltonian and symplectic structure of general isomonodromic deformations we obtained during our studies and check them for Painlevé equations.

1. INTRODUCTION

1.1 Painlevé equations

Differential equations are used to describe various phenomena in engineering, physics, economics and biology. Ordinary differential equations (ODE) describe functions depending only on one variable. The important question is description of singularities of solutions of differential equations. For linear ODEs with rational coefficients the singularities of solutions arise from singularities of coefficients. The local behavior of solutions near corresponding singularities can be described using recursive procedure (see [118]). The relations between behaviors near different singularities for global solution are called connection formulae. For classical special functions they can be obtained using contour integral representations (see [39]). The objective of our work is to find connection formulae in more complicated cases.

For nonlinear ODEs solution can have singularities different from singularities of the coefficients. If for the general solutions position of branching points does not depend on initial conditions, the ODE is said to satisfy Painlevé property. In the case of ODE

$$\frac{d^2u}{dx^2} = F\left(u, \frac{du}{dx}, x\right) \tag{1.1}$$

with right hand side rational in u, $\dfrac{du}{dx}$ and x Painlevé [105] and Gambier [58] listed all equations with Painlevé property (see [66] for detailed exposition of this list).

Only six of them can not be reduced to linear ODEs or solved in terms of elliptic functions. They were called later Painlevé equations. In standard form they are written as

$$\frac{d^2u}{dx^2} = 6u^2 + x, \tag{PI}$$

$$\frac{d^2u}{dx^2} = 2u^3 + xu + \alpha, \tag{PII}$$

$$\frac{d^2u}{dx^2} = \frac{1}{u}\left(\frac{du}{dx}\right)^2 - \frac{1}{x}\left(\frac{du}{dx}\right) + \frac{\alpha u^2}{x} + \frac{\beta}{x} + \gamma u^3 + \frac{\delta}{u}, \tag{PIII}$$

$$\frac{d^2u}{dx^2} = \frac{1}{2u}\left(\frac{du}{dx}\right)^2 + \frac{3}{2}u^3 + 4xu^2 + 2(x^2-\alpha)u + \frac{\beta}{u}, \tag{PIV}$$

$$\begin{aligned}\frac{d^2u}{dx^2} =& \left(\frac{1}{2u} + \frac{1}{u-1}\right)\left(\frac{du}{dx}\right)^2 - \frac{1}{x}\left(\frac{du}{dx}\right)\\ &+ \frac{(u-1)^2}{x^2}\left(\alpha u + \frac{\beta}{u}\right) + \frac{\gamma u}{x} + \frac{\delta u(u+1)}{u-1},\end{aligned} \tag{PV}$$

$$\begin{aligned}\frac{d^2u}{dx^2} =& \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right)\left(\frac{du}{dx}\right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right)\frac{du}{dx}\\ &+ \frac{u(u-1)(u-x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{u^2} + \frac{\gamma(x-1)}{(u-1)^2} + \delta\frac{x(x-1)}{(u-x)^2}\right),\end{aligned} \tag{PVI}$$

In PIII case one also distinguishes three sub-cases

$$\gamma\delta \neq 0 \tag{PIII(D6)}$$

$$\gamma = 0, \quad \alpha\delta \neq 0 \quad \text{or} \quad \delta = 0, \quad \beta\gamma \neq 0 \tag{PIII(D7)}$$

$$\gamma = \delta = 0 \tag{PIII(D8)}$$

The PIII(D7) equation with $\gamma = 0$ and $\delta = 0$ are equivalent to each other through the change of variables (see [80]). Similarly PIII(D8) equation is equivalent to PIII(D6) with $\alpha = 0$ and $\beta = 0$.

The list of necessary conditions required for Painlevé property is called Painlevé test. If the equation passes Painlevé test then the equation is expected to have Lax pair representation and admit explicit solutions. See [30] for Painlevé test applied to various ODEs and PDEs.

The discussion of current progress on analysis of higher degree and higher order analogs of (1.1) is also presented in [30].

The fact that Painlevé equations can not be reduced to previously known functions was shown rigorously. Umemura introduced in [113] the notion of classical function. They are obtained from the constant functions using the following operations

- integration
- differentiation
- sum, difference, product, quotient
- composition with Abelian function
- taking solutions of algebraic equations with coefficients classical functions
- taking solutions of linear differential equations with coefficients classical functions
- taking solutions of algebraic differential equation of first order with coefficients classical functions

Using differential Galois theory it was shown that generic solutions of Painlevé equations can not be expressed in terms of classical functions (see [102, 112, 114, 115, 116, 119, 120, 80]). Non generic solutions can be rational, algebraic or expressed in terms of classical special functions.

Airy functions	(PII)
Bessel functions,	(PIII)
Parabolic cylinder functions and Hermite orthogonal polynomials,	(PIV)
Confluent hypergeometric functions	
and associated Laguerre orthogonal polynomials,	(PV)
Hypergeometric functions and Jacobi orthogonal polynomials,	(PVI)

Such solutions are presented in [97, 98, 47, 88, 92, 39].

The meromorphic nature of solutions of Painlevé equations is described in [59]: solutions of (PI), (PII) and (PIV) are meromorphic in complex plane, solutions of (PIII) are (PV) are meromorphic in complex plane in the variable $t = \ln x$, and solutions of (PVI) admit meromorphic continuation along any path in $\mathbb{C} \setminus \{0,1\}$. One can think of these facts as rigorous statements of Painlevé property. The estimates on the Nevanlinna characteristic corresponding to solutions of (PI) – (PV) also can be found in [59].

Painlevé equations are self-similar reductions of various nonlinear integrable PDEs admitting soliton solutions and therefore they describe nonlinear wave phenomena [1]

Korteweg – de Vries equation, (PI)

modified Korteweg – de Vries equation, (PII)

Sine – Gordon equation, (PIII)

nonlinear Schrödinger equation, (PIV)

Ernst equation, (PV)

three wave resonant interaction equation, (PVI)

Painlevé equations can be interpreted as Newton's law of motion of particle with time-dependent acting force. Actually, if we make change of variables

$$q(t) = u(t) \tag{PI–PII}$$

$$q(t) = \ln\left(u(e^t)\right) \tag{PIII}$$

$$q(t) = \sqrt{u(t)} \tag{PIV}$$

$$q(t) = \ln\left(\frac{\sqrt{u(e^t)} - 1}{\sqrt{u(e^t)} + 1}\right) \tag{PV}$$

$$q(t) = \frac{F\left(\arcsin\left(\sqrt{u(k^{-2})}\right), k\right)}{2K\left(k\right)}, \quad k = \left(\frac{\theta_2(0, e^{-\pi t})}{\theta_3(0, e^{-\pi t})}\right)^2, \tag{PVI}$$

with $K(k)$ and $F(\phi,k)$ complete and incomplete elliptic integrals of the first kind, then the equations of motion are given by

$$\frac{d^2q}{dt^2}=6q^2+t \tag{PI(F)}$$

$$\frac{d^2q}{dt^2}=2q^3+qt+\alpha, \tag{PII(F)}$$

$$\frac{d^2q}{dt^2}=\alpha e^{t+q}+\beta e^{t-q}+\gamma e^{2t+2q}+\delta e^{2t-2q}, \tag{PIII(F)}$$

$$\frac{d^2q}{dt^2}=\frac{3q^5}{4}+2tq^3+q(t^2-\alpha)+\frac{\beta}{2q^3}, \tag{PIV(F)}$$

$$\frac{d^2q}{dt^2}=-\frac{\alpha\cosh\left(\frac{q}{2}\right)}{\sinh^3\left(\frac{q}{2}\right)}-\frac{\beta\sinh\left(\frac{q}{2}\right)}{\cosh^3\left(\frac{q}{2}\right)}-\frac{\gamma}{2}e^t\sinh(q)-\frac{\delta}{4}e^{2t}\sinh(2q), \tag{PV(F)}$$

$$\frac{d^2q}{dt^2}=\frac{4K^3}{\pi^2}\left[\alpha k^2\mathrm{sn}(2qK,k)\mathrm{cn}(2qK,k)\mathrm{dn}(2qK,k)+\beta\frac{\mathrm{cn}(2qK,k)\mathrm{dn}(2qK,k)}{(\mathrm{sn}(2qK,k))^3}+\right.$$
$$\left.+\gamma(1-k^2)\frac{\mathrm{sn}(2qK,k)\mathrm{dn}(2qK,k)}{(\mathrm{cn}(2qK,k))^3}+\left(\delta-\frac{1}{2}\right)k^2(1-k^2)\frac{\mathrm{sn}(2qK,k)\mathrm{cn}(2qK,k)}{(\mathrm{dn}(2qK,k))^3}\right]. \tag{PVI(F)}$$

The changes of variables described above are mentioned in [109, 3, 90]. The interpretation as equation of motion helps better understand qualitatively the behavior of solutions, as we will see below.

The other physical and mathematical applications of Painlevé equations include backward Kolmogorov equation [107, 13, 29], nonintersecting particle systems [25], fluid dynamics [81, 57], wave scattering [99], black hole scattering [103], Rabi model [31], Hele-Shaw process [50], two-dimensional quantum gravity [60, 24, 41], quantum cohomology and topological quantum field theory [42, 61], conformal field theory [55, 56, 9, 100], gauge theory [19, 18], impenetrable Bose gas [77], holonomic quantum fields [108], Ising model [7], spin chains [111], Glauber-Ising chain [40], combinatorics [43, 5], topological recursion [74], orthogonal polynomials [117], differential geometry of surfaces [16], gap probabilities in random matrix theory [51].

Simultaneously with the appearance of the Painlevé transcendents in the numerous applications, their analytical theory has been dramatically developed during the last 40 year. One of the main reasons why Painlevé functions have been studied so well is that they describe isomonodromic deformations of the linear systems of ODEs with rational coefficients and hence admit the so-called *Riemann-Hilbert Representation.* This allows, in particular, to perform global asymptotic analysis of the Painlevé transcendents including explicit evaluation of the *connection formulae* between the asymptotic parameters of the Painlevé functions at different relevant critical points. In the next two Sections we will outline the basic ideas and demonstrate the effectiveness of the Riemann-Hilbert method by presenting some of the known results concerning the Painlevé II and Painlevé VI equations.

1.1.1 Pure imaginary solution of homogeneous PII equation

Consider purely imaginary solutions of PII equation with $\alpha = 0$. This equation is also called homogeneous PII equation. To describe it as isomonodromic deformation, consider the following 2×2 linear system of ODE's with one irregular singular point of Poincaré rank 3 at infinity

$$\frac{d\Phi}{dz} = A\left(z\right)\Phi, \qquad A\left(z\right) = A_{-3}z^2 + A_{-2}z + A_{-1}. \tag{1.2}$$

$$A_{-3} = -4i\sigma_3, \quad A_{-2} = -4q\sigma_2, \quad A_{-1} = (-it - 2iq^2)\sigma_3 - 2p\sigma_1$$

where Pauli matrices σ_1, σ_2, σ_3 are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Seven canonical solutions of (1.2) are uniquely specified by the following asymptotic conditions

$$\Phi_j\left(z\right) \simeq \left(I + \sum_{m=1}^{\infty} g_m z^{-m}\right) e^{-\left(\frac{4i}{3}z^3 + itz\right)\sigma_3}, \quad z \to \infty, \quad z \in \Omega_j, \quad j = 1, \dots, 7, \tag{1.3}$$

where the Stokes sectors are given by

$$\Omega_j = \left\{ z : \frac{\pi(j-2)}{3} < \arg z < \frac{\pi j}{3} \right\}.$$

There are six Stokes matrices $S_1, \dots, S_6$ defined by the equations

$$S_j = \Phi_j^{-1}\left(z\right)\Phi_{j+1}\left(z\right), \qquad j = 1, \dots, 6.$$

These matrices have the familiar triangular structure (see [49, chapter 2, section 1.6]).

$$S_{2l+1} = \begin{pmatrix} 1 & 0 \\ s_{2l+1} & 1 \end{pmatrix}, \quad S_{2l} = \begin{pmatrix} 1 & s_{2l} \\ 0 & 1 \end{pmatrix}, \tag{1.4}$$

For $j = 1, \ldots, 6$, let Γ_j denote the rays

$$\Gamma_j = \left\{ z \in \mathbb{C} : \arg z = \frac{\pi(2j-1)}{6} \right\}.$$

oriented towards infinity, and let $\Omega_j^{(0)}$ be the sectors between the rays Γ_{j-1} and Γ_j. Note that $\overline{\Omega_j^{(0)}} \subset \Omega_j$.

We can notice the symmetry

$$-A(-z) = \sigma_2 A(z) \sigma_2$$

which implies

$$\Phi(-z) = \sigma_2 \Phi(z) \sigma_2, \quad S_{n+3} = \sigma_2 S_n \sigma_2, \quad s_{l+3} = -s_l. \tag{1.5}$$

The fact that $q(t)$ is purely imaginary solution implies the symmetry

$$\overline{A(\overline{z})} = \sigma_2 A(z) \sigma_2. \tag{1.6}$$

which gives

$$\overline{\Phi(\overline{z})} = \sigma_2 \Phi(z) \sigma_2, \quad \left(\overline{S_n}\right)^{-1} = \sigma_2 S_{7-n} \sigma_2, \quad s_{7-l} = \overline{s_l}. \tag{1.7}$$

Canonical solutions satisfy monodromy condition

$$\Phi_7(z) = \Phi_1(z).$$

It implies the cyclic relation

$$S_1 S_2 S_3 S_4 S_5 S_6 = I. \tag{1.8}$$

which can be written as scalar equation taking into account (1.5)

$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0. \tag{1.9}$$

Using that and (1.7) we get

$$s_2 = \frac{2i \mathrm{Im} s_1}{1 + |s_1|^2}, \quad s_3 = -\overline{s_1}.$$

Therefore Stokes matrices corresponding to pure imaginary solutions $q(t)$ are parametrized by one complex parameter s_1.

Define a piecewise analytic function $\Psi(z)$ by the relations

$$\Psi(z) = \Phi_j(z) \qquad \text{for} \;\; z \in \Omega_j^{(0)}. \tag{1.10}$$

The function $\Psi(z)$ satisfies the following Riemann-Hilbert problem posed on the contour $\Gamma = \bigcup_{j=1}^{6} \Gamma_j$:

Riemann-Hilbert Problem 1.1.

- $\Psi(z)$ *is analytic for* $z \in \mathbb{C} \setminus \Gamma$,
- $\Psi_+(z) = \Psi_-(z) J(z)$ *for* $z \in \Gamma$,
- $\Psi(z)$ *satisfies condition* (1.3) *at infinity.*

The contour Γ *and the associated piecewise constant jump matrices* $J(z)$ *are depicted in Figure 1.1.*

Here "+" refers to the boundary value from the left of the contour Γ and "−" refers to the boundary value from the right of the contour Γ.

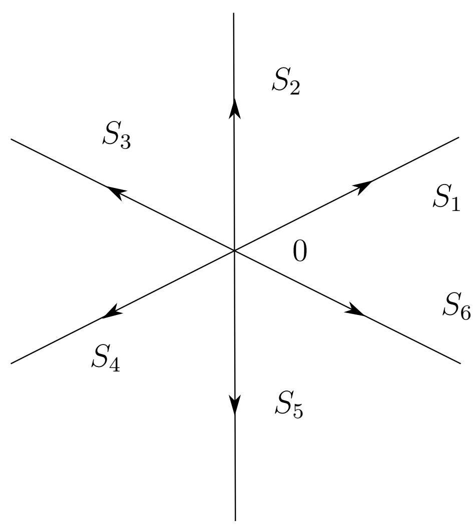

Figure 1.1. Contour Γ and jump matrices $J(z)$ for the RHP 1.1

Solution of the Riemann-Hilbert problem is uniquely determined by the parameter t and Stokes matrices. If we fix Stokes matrices, the dependence of $\Psi(z)$ on t is describing isomonodromic deformation. We have the equation

$$\frac{d\Psi}{dt} = U(z)\Psi, \qquad U(z) = -iz\sigma_3 - q\sigma_2. \tag{1.11}$$

In the same time $\Psi(z)$ satisfies (1.2) since it is given by (1.10). The compatibility condition of (1.2) and (1.11) is given by

$$\frac{dA}{dt} - \frac{dU}{dz} + [A, U] = 0,$$

or in scalar form

$$\frac{dp}{dt} = 2q^3 + qt, \quad \frac{dq}{dt} = p.$$

This is equivalent to PII equation for $q(t)$. It also can be described by the formula,

$$q = 2\,(g_1)_{12}\,,$$

where g_1 is the first matrix coefficient in the asymptotic expansion (1.3).

We denote $q(t) = iy(t)$. The equation of motion for real valued solution $y(t)$ has acting force

$$F(y,t) = -2y^3 + yt.$$

We present the vector field graph of the acting force on Figure 1.2. We can see that there are two stable trajectories $y = \pm\sqrt{\dfrac{t}{2}}$ for positive time and trajectory $y = 0$, which is stable for negative time and unstable for positive time.

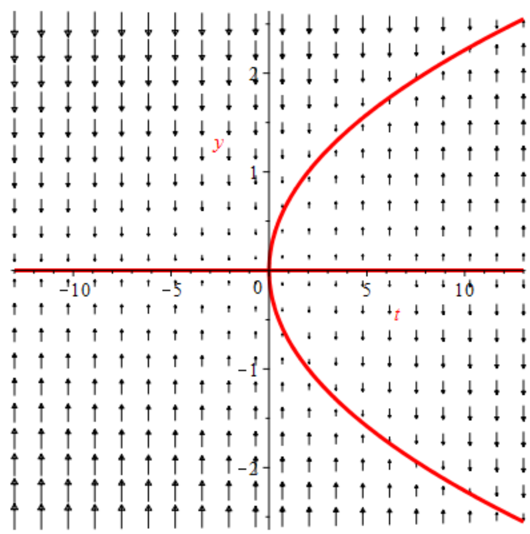

Figure 1.2. Force field for pure imaginary solutions of homogeneous PII equation

Finding approximation for large $t \to \pm\infty$ for solutions $\Psi(z,t)$ of the Riemann-Hilbert problem allows to obtain the following description approximations for $y(t)$ (see [68], [38])

$$y(t) = \frac{d}{(-t)^{\frac{1}{4}}} \sin\left(\frac{2}{3}(-t)^{\frac{3}{2}} + \frac{3}{4}d^2\ln(-t) + \phi\right) + O\left(\frac{1}{|t|}\right), \quad t \to -\infty,$$

$$y(t) = \sigma\sqrt{\frac{t}{2}} + \frac{\sigma\rho}{(2t)^{\frac{1}{4}}} \cos\left(\frac{2\sqrt{2}}{3}t^{\frac{3}{2}} - \frac{3}{2}\rho^2\ln t + \theta\right) + O\left(\frac{1}{t}\right), \quad t \to +\infty.$$

where

$$d = \sqrt{\frac{1}{\pi} \ln\left(1 + |s_1|^2\right)}, \quad \phi = -\frac{\pi}{4} + \frac{3}{2} d^2 \ln 2 - \arg\left(\Gamma\left(i\frac{d^2}{2}\right)\right) - \arg(s_1). \tag{1.12}$$

$$\rho = \sqrt{\frac{1}{\pi} \ln\left(\frac{1 + |s_1|^2}{2|\mathrm{Im}(s_1)|}\right)}, \quad \sigma = -\mathrm{sign}(\mathrm{Im}(s_1)), \tag{1.13}$$

$$\theta = -\frac{3\pi}{4} - \frac{7}{2}\rho^2 \ln 2 + \arg(\Gamma(i\rho^2)) + \arg(1 + s_1^2). \tag{1.14}$$

The formula at $t \to +\infty$ is valid for $\mathrm{Im}(s_1) \neq 0$. If $s_1 = \pm i$ then $\rho = 0$ and the asymptotic has form of power series. The values $\mathrm{Im}(s_1) = 0$ correspond to the separatrix case – the Ablowitz-Segur solution, whose behavior at $t \to +\infty$ is replaced by

$$y(t) = \frac{s_1}{2\sqrt{\pi} t^{\frac{1}{4}}} e^{-\frac{2}{3} t^{\frac{3}{2}}} \left(1 + O\left(\frac{1}{t^{\frac{3}{4}}}\right)\right), \quad t \to +\infty.$$

One can solve equations (1.12) for s_1,

$$|s_1| = \left(e^{\pi d^2} - 1\right)^{1/2}, \quad \arg s_1 = -\phi - \frac{\pi}{4} + \frac{3}{2} d^2 \ln 2 - \arg\left(\Gamma\left(i\frac{d^2}{2}\right)\right),$$

and then equations (1.13), (1.14) gives the explicit *connection formulae* between the asymptotic parameters (d, ϕ) at $t = -\infty$ and the asymptotic parameters (ρ, θ, σ) at $t = +\infty$.

The connection formulae allow to determine if the particle at positive time will end up near trajectory $y = \sqrt{\frac{t}{2}}$, $y = -\sqrt{\frac{t}{2}}$ or $y = 0$ based on the phase of particle oscillation at negative time. It turns out that smallest change in phase oscillation at negative infinity can change trajectory at positive infinity drastically (see Figure 1.3). Explicit description of such global properties of solutions is usually available for equations coming from integrable systems and not for general ODEs.

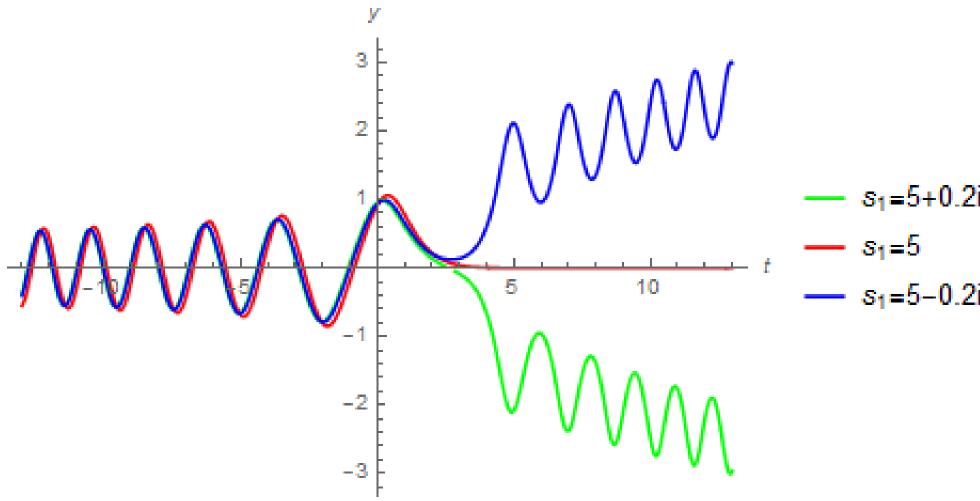

Figure 1.3. Different behaviors of pure imaginary solutions of homogeneous PII equation

1.1.2 General solution of PVI equation

Consider another example of general solution of Painlevé VI equation. We look at the Fuchsian system with 4 regular singularities at $0, 1, x$ and ∞

$$\frac{d\Phi}{dz} = A(z)\,\Phi, \qquad A(z) = \frac{A_0}{z} + \frac{A_x}{z-x} + \frac{A_1}{z-1}. \tag{1.15}$$

Assume that

$$A_0 + A_x + A_1 = -\frac{\theta_\infty}{2}\sigma_3, \quad \pm\frac{\theta_\nu}{2} - \text{eigenvalues of } A_\nu, \quad \nu = 0, x, 1.$$

Relation between parameters $\theta_0, \theta_1, \theta_x, \theta_\infty$ and $\alpha, \beta, \gamma, \delta$ is given by

$$\alpha = \frac{(\theta_\infty - 1)^2}{2}, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = \frac{\theta_1^2}{2}, \quad \delta = \frac{1-\theta_x^2}{2},$$

$$\operatorname{Re}\theta_\nu \geq 0, \quad \nu = 0, 1, x, \quad \operatorname{Re}\theta_\infty \geq 1.$$

We have four solutions of (1.15) described by convergent series

$$\begin{aligned}
\Phi^{(\nu)}(z) &= G_\nu \left[I + \sum_{m=1}^{\infty} g_{\nu,m}(z-\nu)^m\right](z-\nu)^{\frac{\theta_\nu}{2}\sigma_3}(z-\nu)^{R_\nu}, \quad |z-\nu| < r, \\
\Phi^{(\infty)}(z) &= \left[I + \sum_{m=1}^{\infty} g_{\infty,m} z^{-m}\right] z^{-\frac{\theta_\infty}{2}\sigma_3} z^{-R_\infty}, \quad |z| > R.
\end{aligned} \tag{1.16}$$

Here matrices G_ν are matrices putting A_ν in Jordan form. Matrices R_ν are given by

$$R_\nu = \begin{cases} \begin{pmatrix} 0 & r_\nu \\ 0 & 0 \end{pmatrix}, & \text{if } \theta_\nu > 0,\ \theta_\nu \in \mathbb{Z}, \\ \begin{pmatrix} 0 & 0 \\ r_\nu & 0 \end{pmatrix}, & \text{if } \theta_\nu < 0,\ \theta_\nu \in \mathbb{Z}, \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \text{if } \theta_\nu = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad \nu = 0, 1, x, \infty.$$

The solutions $\Phi^{(\nu)}$ are related to each other using connection matrices C_ν.

$$\Phi^{(\infty)}(z) = \Phi^{(\nu)}(z) C_\nu.$$

We introduce matrices M_ν by

$$M_\infty = e^{i\pi\theta_\infty\sigma_3} e^{2\pi i R_\infty}, \quad M_\nu = C_\nu^{-1} e^{i\pi\theta_\nu\sigma_3} e^{2\pi i R_\nu} C_\nu, \quad \nu = 0, 1, x,$$

They are counterclockwise monodromy matrices around ν obtained along the loops on Figure 1.4 (z_1 is the base point). They satisfy the condition

$$M_1 M_x M_0 M_\infty = I, \tag{1.17}$$

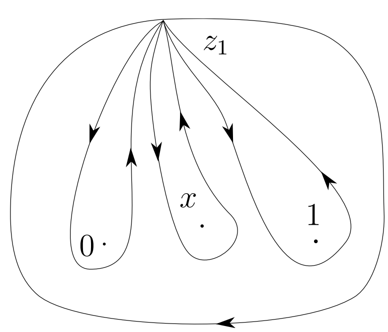

Figure 1.4. Choice of loops for counterclockwise monodromy matrices M_ν for PVI equation.

Using functions $\Phi^{(\nu)}$ we can construct solution of the following Riemann-Hilbert problem

Riemann-Hilbert Problem 1.2.

- $\Psi(z)$ *is analytic for* $z \in \mathbb{C} \setminus \Gamma$*;*
- $\Psi_+(z) = \Psi_-(z)J(z)$ *for* $z \in \Gamma$*,*
- *Near the points* $0, 1, x, \infty$ *the behavior of* $\Psi(z)$ *is described by* (1.16).

Contour Γ *and jump matrices* $J(z)$ *are shown on the Figure 1.5 (*z_0 *is some reference point).*

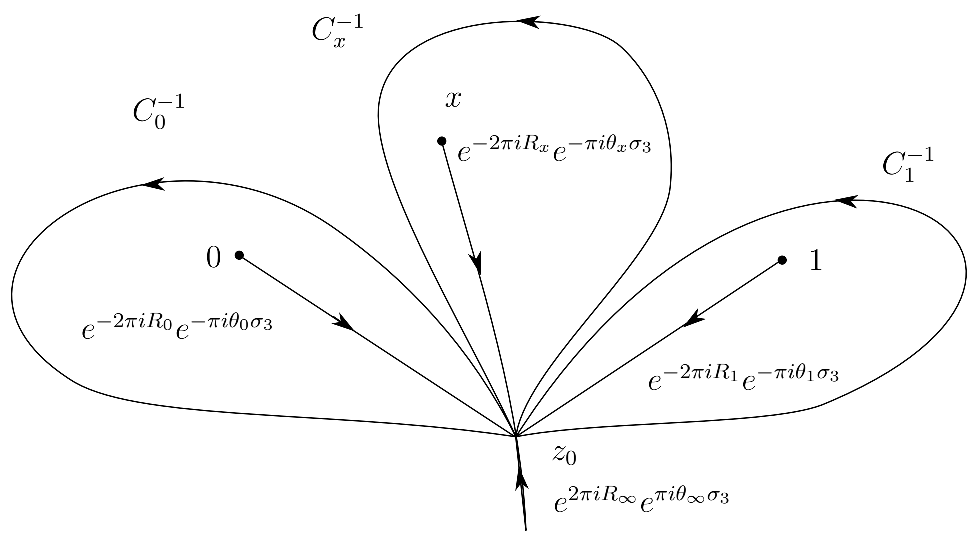

Figure 1.5. Contour Γ and jump matrices $J(z)$ for the RHP 1.2

The solution $\Psi(z)$ is uniquely determined by $x, C_\nu, \theta_\nu, r_\nu$. If we fix C_ν, θ_ν, r_ν and start changing x, then such deformation is called isomonodromic. In this case $\Psi(z)$ satisfies the equation.

$$\frac{d\Psi}{dx} = -\frac{A_x}{z-x}\Psi.$$

It implies the following equations

$$\frac{dG_0}{dx} = \frac{A_x}{x}G_0, \quad \frac{dG_1}{dx} = \frac{A_x}{x-1}G_1, \quad \frac{dG_x}{dx} = \left(\frac{A_0}{x} + \frac{A_1}{x-1}\right)G_x,$$

Then the solution of PVI equation is given by

$$u(x) = \frac{x(A_0)_{12}}{x((A_0)_{12} + (A_1)_{12}) + (A_0)_{12} + (A_x)_{12}}.$$

Consider the following trace functions

$$p_\nu = \mathrm{Tr}(M_\nu), \quad p_{\mu\lambda} = \mathrm{Tr}(M_\mu M_\lambda), \quad \nu = 0, 1, x, \quad \mu, \lambda \in \{0, 1, x\}.$$

They satisfy the Jimbo-Fricke relation, which is the trace of (1.17)

$$p_{0x}^2 + p_{01}^2 + p_{x1}^2 + p_{0x}p_{01}p_{x1} - (p_0p_x + p_1p_\infty)p_{0x} - (p_0p_1 + p_xp_\infty)p_{01} - (p_xp_1 + p_0p_\infty)p_{x1} +$$

$$+p_0^2 + p_1^2 + p_x^2 + p_\infty^2 + p_0p_xp_1p_\infty - 4 = 0 \tag{1.18}$$

Now to describe all branches of solutions of PVI equation we need to consider two subsets of solutions.

If M_0, M_1, M_x generate irreducible representation of $\pi_1(\mathbb{C}/\{0, 1, x\})$ then it is shown in [75] that corresponding solutions of PVI equation are parametrized with coordinates

$$\{p_{0x}, p_{01}, p_{x1} : (1.18) \text{ holds}\} \tag{1.19}$$

If M_0, M_1, M_x generate reducible representation of $\pi_1(\mathbb{C}/\{0, 1, x\})$, then they correspond to critical points of (1.18) and it was shown in [64, 92] that in this case corresponding solutions of PVI equation are rational or expressed in terms of hypergeometric functions.

All asymptotic behaviors of PVI solutions as $x \to 0, 1, \infty$ corresponding to all possible values of parameters (1.19), including the critical points, are described in [62].

The real solutions with no singularities on the interval $[1, \infty)$ was considered in [46]. Among the monodromy matrices M_ν in this case there are 2 matrices with common eigenvector. Up to bi-rational transformations, such solutions have asymptotics (41) – (46) in [62]. It should be also mentioned that the authors of [46] has developed an algorithm to compute the numbers of zeros, poles, 1-points and fixed points of a real solution of Painlevé VI equation on the interval $(1, +\infty)$. and their mutual position. This algorithm is based on a remarkable link, also discovered in [46], between the real solutions of Painlevé VI equations and a special geometric object, a one parameter family of circular pentagons.

1.2 The Painlevé tau functions

The notion of tau function was introduced in the theory of system of linear ordinary differential equations by Jimbo, Miwa, and Ueno in the 80s ([78]), and it has gradually become one of the central concepts not only in the theory of linear differential equations but in the whole general area of integrable systems and their applications, especially in the problems related to random matrices and statistical mechanics [52, 53, 54, 82, 34]. The exact definition of the object will be presented in Section 2.1. Here, we only mention that, in the case of Painlevé equations, a characteristic property of the tau function of a given Painlevé equation is that its logarithmic derivative is the Hamiltonian of the equation.

Usually, it is not the Painlevé functions *per se* but the related tau functions that are objects which actually appear in applications, notably in the description of the correlation functions of integrable statistical mechanics and quantum field models. The main analytic issue in these applications is the behavior of the tau functions near the relevant fixed critical points. Of the special importance is the evaluation of the connection formulae between the corresponding asymptotic parameters including the evaluation of the constant pre-factors appearing in the asymptotics. The latter, very often, contain the most important information of the models in question.

As it has already been indicated in the last paragraph before Section 1.1.1 of this introduction, one of the principal achievements of modern theory of the Painlevé transcendents is that it is possible to produce connection formulae for these functions in closed form. However, this fact by itself does not solve the connection problem for the associated tau functions. Indeed, as it was mentioned above, the defining property of the tau function of a given Painlevé equation is that its logarithmic derivative is the Hamiltonian of the equation. Hence, in order to obtain the full connection formulae for Painlevé tau functions one should be able to evaluate integrals of certain combinations of Painlevé transcendents and their derivatives – the combinations which make the corresponding Hamiltonians. In other words, a complete connection formula for a given tau function must include the precise information of certain constant of integrations (or, rather their ratios at different critical points) which manifest themselves as pre-factors in the tau function expansions. The evaluation of these integration constants is often refer to as a "constant problem". After the development in the 90s of the efficient techniques of the asymptotic evaluation of the Painlevé functions *per se* ([38, 37]), the constant problem has been a major challenge in the asymptotic study of Painlevé equations.

The first rigorous solution of a constant problem for Painlevé equations was obtained in the 1991 work of Tracy [110] where he studied a special solution of the third Painlevé equation in the context of the phase transition in the 2D Ising model. We describe the setting of this problem below.

1.2.1 Tau function for special solution of PIII(D8)(F) equation

Consider symmetric Ising model on $\mathbb{Z}^2$. It describes phase transition in ferromagnetism. The configuration σ represents the spin orientation at every point on the integer lattice. It is given by the function

$$\sigma : \mathbb{Z}^2 \to \{1, -1\}.$$

Energy of configuration σ restricted to $M \times N$ rectangle $\Lambda \in \mathbb{Z}^2$ is defined by the formula

$$E_\Lambda(\sigma) = -J \sum_{j,k\in\Lambda} (\sigma_{j,k}\sigma_{j,k+1} + \sigma_{j,k}\sigma_{j+1,k}), \quad J > 0.$$

Introduce spin correlation function along the row

$$\langle \sigma_{1,1}\sigma_{1,n+1} \rangle = \lim_{|\Lambda|\to\infty} \frac{\sum\limits_{\sigma} \sigma_{1,1}\sigma_{1,n+1} e^{-\frac{E_\Lambda(\sigma)}{kT}}}{\sum\limits_{\sigma} e^{-\frac{E_\Lambda(\sigma)}{kT}}}.$$

Introduce the notation

$$z = \tanh\left(\frac{J}{kT}\right).$$

The critical temperature T_c corresponds to the value

$$z_c = \sqrt{2} - 1.$$

The following results one can find in [95]. For $T < T_c$ the correlation function exhibits long range interaction

$$\lim_{n\to\infty} \langle \sigma_{1,1}\sigma_{1,n+1} \rangle = M_0 = \left[\frac{(z^2+1)^2(4z^2-(z^2-1)^2)}{16z^4}\right]^{\frac{1}{4}}. \tag{1.20}$$

Parameter M_0 is called spontaneous magnetization. For $T = T_c$ the correlation function decays powerlike

$$\langle \sigma_{1,1}\sigma_{1,n+1}\rangle \simeq c_1 n^{-\frac{1}{4}}, \quad n\to\infty, \quad c_1 = e^{\frac{1}{4}}A^{-3}2^{\frac{5}{24}}. \tag{1.21}$$

For $T > T_c$ the correlation function decays exponentially

$$\begin{gathered}\langle \sigma_{1,1}\sigma_{1,n+1}\rangle \simeq c_2\left(\frac{e^{-n\ln c_3}}{\sqrt{\pi n}}\right), \quad n\to\infty,\\ c_2 = \left[\frac{(z^2-1-2z)(z^2-1)^2}{(z^2-1+2z)16z^2}\right]^{\frac{1}{4}}, \quad c_3 = \frac{1-z}{z(1+z)}.\end{gathered} \tag{1.22}$$

We see that after the temperature surpasses critical value, the spins become much less correlated. To describe the phase transition between (1.20), (1.21), and (1.22) we introduce the correlation length $\zeta(T)$ by formula

$$\zeta(T) = \frac{\sqrt{z(1-z^2)}}{|z^2+2z-1|}.$$

When the temperature approaches critical value, the correlation length approaches to infinity and the correlation function approaches to zero

$$\zeta(T) \simeq \frac{1}{2(\sqrt{2}+1)(z-z_c)}, \quad M_0 \simeq 2^{\frac{5}{8}}(\sqrt{2}+1)^{\frac{1}{4}}(z-z_c)^{\frac{1}{4}}, \quad z\to z_c.$$

This behavior motivates to consider the following double-scaling limit

$$T\to T_c, \quad n\to\infty, \quad \exp\left(\frac{t}{2}\right) = \frac{n}{\zeta(T)} - \text{fixed}.$$

The following formula holds [6]

$$\lim_{\substack{n\to\infty\\ T\to T_c\pm 0}} n^{\frac{1}{4}}\langle \sigma_{1,1}\sigma_{1,n+1}\rangle = 2^{\frac{3}{8}}e^{\frac{t}{8}}\exp\left(-\int\limits_t^{+\infty}\left(\frac{H}{4}+\frac{e^t}{16}\right)dt\right)\begin{cases}\sinh(\frac{q}{4}),\ T > T_c,\\ \cosh(\frac{q}{4}),\ T < T_c.\end{cases} \tag{1.23}$$

The Hamiltonian H here is given by

$$H(p,q,t) = \frac{p^2}{2} - \frac{e^t\cosh(q)}{4}, \quad p = \frac{dq}{dt}.$$

The integral appearing in (1.23) is related to the tau function

$$\tau(t_1,t_2) = \exp\left(\int\limits_{t_1}^{t_2} H dt\right). \tag{1.24}$$

The function $q(t)$ solves PIII(D8)(F) equation with $\alpha = \frac{1}{8}, \quad \beta = -\frac{1}{8}, \quad \gamma = \delta = 0$

$$\frac{d^2 q}{dt^2} = \frac{1}{4} e^t \sinh q. \tag{1.25}$$

The plot of corresponding vector field is given on Figure 1.6.

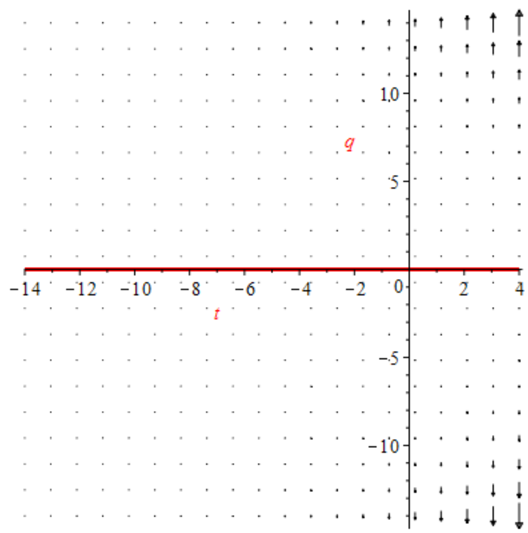

Figure 1.6. Force field for PIII(D8)(F) equation (1.25)

We can express $q(t)$ in notations of [6, 94]

$$q(t) = 2\psi\left(\exp\left(\frac{t}{2}\right)\right) = -2\ln\eta\left(\frac{1}{2}\exp\left(\frac{t}{2}\right)\right)$$

and in notations of [101, 72]

$$q(t) = -iu\left(e^{\frac{t}{2}}\right) + i\pi.$$

The asymptotic of the solution $q(t)$ appearing in (1.23) is given by (see Figure 1.7)

$$q(t) \simeq -t + 4\ln(2) - 2\ln(6\ln(2) - 2\gamma - t), \quad t \to -\infty, \tag{1.26}$$

$$q(t) \simeq 2\sqrt{\frac{2}{\pi}} e^{-\frac{t}{4}} e^{-e^{\frac{t}{2}}}, \quad t \to +\infty. \tag{1.27}$$

where γ is the Euler's constant. It was rigorously justified in [121, 101].

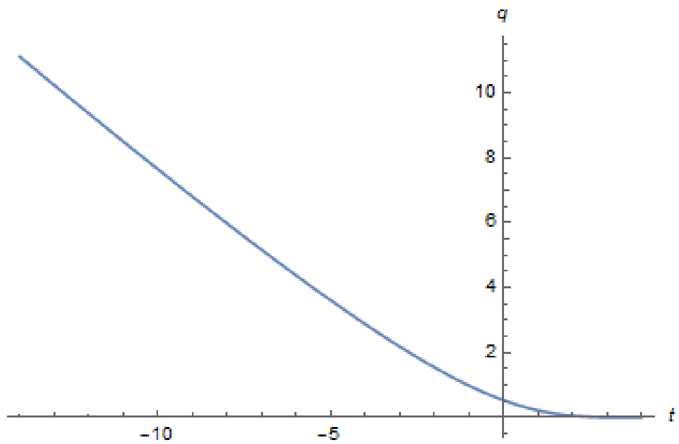

Figure 1.7. Solution of PIII(D8)(F) equation (1.25) with asymptotics (1.26), (1.27)

To solve the connection problem for tau function means to compute the asymptotic $t \to -\infty$ of the integral (1.23). As it was mentioned earlier, it was done by Tracy in [110] and the answer is

$$\exp\left(-\int_t^{+\infty}\left(\frac{H}{4}+\frac{e^t}{16}\right)dt\right)\frac{e^{\frac{q}{4}}}{2} \simeq e^{-\frac{t}{8}}A^{-3}e^{\frac{1}{4}}2^{-\frac{1}{6}}, \quad t \to -\infty \tag{1.28}$$

where A is Glaisher-Kinkelin constant. Plugging it in (1.23) we get the correct value of constant c_1 in (1.21). That verifies that the description of transition between (1.20), (1.21), and (1.22) is given by tau function (1.24). The proof of (1.28) is based on approximation of the solution (1.26), (1.27) by family of solutions

$$q(t) \simeq at - 6a\ln(2) + 2\ln\frac{\Gamma\left(\frac{1-a}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)}, \quad t \to -\infty, \tag{1.29}$$

$$q(t) = -2\sin\left(\frac{\pi a}{2}\right)\sqrt{\frac{2}{\pi}}\exp\left(-\frac{t}{4}\right)\exp\left(-e^{\frac{t}{2}}\right)(1+0(1)), \quad t \to +\infty.$$

with $-1 < a < 0$.

Another approach using approximation of solutions of PIII(D8) equation with family of solutions PIII(D6) equations is presented in [94, 23]. The analysis of phase transition above in terms of appearance of Fisher-Hartwig singularity in Toeplitz determinant is given in [28].

1.3 The goals and the outline of the dissertation

A number of other constant problems for Painlevé tau functions were solved in recent years [8, 85, 33, 4, 35, 44, 83, 36, 45]. The tau functions that appeared in all these papers, however, corresponded to very special families of Painlevé functions with some extra, non-generic properties which were "responsible" for the solution of the problem. The explicit evaluation of the constant pre-factors in the tau function expansions for generic Painlevé transcendents has been *one of the top open questions* in the area since the above mentioned paper of Craig Tracy [110]. Only very recently, due to the important works of Gamayun, Iorgov, Lisovyy on conformal field theory interpretation of the Painlevé tau-functions [55, 56], a progress in the "constant problem" began to appear. However, the results obtained in [67, 70] though very important have not been mathematically rigorous.

A main goal of this dissertation is the rigorous evaluation of the constant pre-factors for generic families of tau functions for Painlevé II and III(D8)(F) equations following [72, 69, 73]. It boils down to Theorems 4.1 and 4.2. We shall conclude this introduction by outlining the content of the dissertation in more details.

We derive in Chapter 2 the differential identities for general isomonodromic tau function which allow to reduce the connection problem for tau function to the connection problem for solutions of isomonodromic deformation equations (see Theorem 2.1). For Painlevé equations they are the consequence of quasihomogeneity of Hamiltonians as we show in Chapter 3 (see Theorem 3.1). We use the identities from Chapter 3 to solve connection problem for tau function for generic solutions of PIII(D8)(F) equation in Section 4.1 and homogeneous PII equation in Section 4.2.

During the derivation of differential identities mentioned above we arrived to Conjectures 2.1, 2.2, 5.1, and 5.2 on the Hamiltonian and symplectic structure of general isomonodromic deformations. We formulate them in Chapters 2 and 5 and check for Painlevé equations in Chapter 6.

We also would like to mention recent works on connection problems for tau functions [22, 20, 23, 87, 86, 93, 32] which appeared after [72, 69, 73].

2. ISOOMONODROMIC TAU FUNCTION

2.1 Isomonodromic deformations

In this chapter we give the definition of isomonodromic tau function and derive the differential identities for it. Consider the system of linear differential equations with rational coefficients with $n+1$ singularities at $a_1, \ldots, a_n, a_\infty = \infty$ on $\hat{\mathbb{C}}$

$$\frac{d\Phi}{dz} = A(z)\,\Phi, \qquad A(z) = \sum_{\nu=1}^{n}\sum_{k=1}^{r_\nu+1} \frac{A_{\nu,-k+1}}{(z-a_\nu)^k} - \sum_{k=0}^{r_\infty-1} z^k A_{\infty,-k-1}. \tag{2.1}$$

Without loss of generality we assume that

$$A_{\nu,-k+1}, A_{\infty,-j-1} \in \mathfrak{sl}_N(\mathbb{C}), \quad k = 1\ldots r_\nu+1, \quad j = 0\ldots r_\infty - 1, \quad \nu = 1\ldots n$$

We shall also assume that all highest order matrix coefficients $A_{\nu,-r_\nu}$ are diagonalizable

$$A_{\nu,-r_\nu} = G_\nu \Theta_{\nu,-r_\nu} G_\nu^{-1}; \quad \Theta_{\nu,-r_\nu} = \operatorname{diag}\{\theta_{\nu,1}, \ldots \theta_{\nu,N}\},$$

and that their eigenvalues are distinct and non-resonant:

$$\begin{cases} \theta_{\nu,\alpha} \neq \theta_{\nu,\beta} & \text{if} \quad r_\nu \geq 1, \quad \alpha \neq \beta, \\ \theta_{\nu,\alpha} \neq \theta_{\nu,\beta} \mod \mathbb{Z} & \text{if} \quad r_\nu = 0, \quad \alpha \neq \beta. \end{cases}$$

Matrices G_ν are determined up to right multiplication by diagonal matrices. We make $\det(G_\nu) = 1$ and keep other $N-1$ parameters free. Using the transformation $\Phi \to C\Phi$ with constant matrix C we get $A_{\infty,-r_\infty}$ diagonal. If $r_\infty = 0$, then we define

$$A_{\infty,0} = -\sum_{\nu=1}^{n} A_{\nu,0}.$$

and make it diagonal. In other words, we assume $G_\infty = I$.

We introduce the space $\mathcal{A}$ of coefficients of (2.1).

$$\mathcal{A} = \{a_\nu \in \mathbb{C},\ A_{\nu,-k+1},\ A_{\infty,-j-1},\ \Theta_{\nu,-r_\nu},\ \Theta_{\infty,-r_\infty} \in \mathfrak{sl}_N(\mathbb{C}),\ G_\nu \in SL_N(\mathbb{C}),$$
$$k = 1\ldots r_\nu,\ j = 0\ldots r_\infty - 2,\ \nu = 1\ldots n\}/\sim$$

Two extra constraints are put using change of variable $z \to \alpha z + \beta$. As the result we have the following formula for dimension of $\mathcal{A}$

$$\begin{aligned}\dim\mathcal{A} &= n + (N^2-1)\left(\sum_{\nu=1}^{n} r_\nu + r_\infty - 1\right) + (N-1)(n+1) + n(N^2-1) - 2 \\ &= \left(\sum_{\nu=1}^{n} r_\nu + r_\infty\right)(N^2-1) + (N-1)(n+1) + (N^2-1)(n-1) + n - 2.\end{aligned} \tag{2.2}$$

If the *Poincaré index* r_ν of the pole a_ν is greater or equal to 1, then the pole is called an *irregular singular point* of the system (2.1). In the neighborhood of such a point the asymptotic behavior of solution $\Phi(z)$ exhibits the *Stokes Phenomenon* which is described as follows.

Let a_ν be an irregular singular point of index r_ν. For $j = 1, \ldots, 2r_\nu + 1$, let

$$\Omega_{j,\nu} = \left\{ z : 0 < |z - a_\nu| < \epsilon, \quad \theta_j^{(1)} < \arg(z - a_\nu) < \theta_j^{(2)}, \quad \theta_j^{(2)} - \theta_j^{(1)} = \frac{\pi}{r_\nu} + \delta \right\},$$

be the *Stokes sectors* around a_ν (see, e.g., [49, Chapter 1] and [118] for more details). According to the general theory of linear systems, in each sector $\Omega_{j,\nu}$ there exists a unique *canonical solution* $\Phi_j^{(\nu)}(z)$ of (2.1) which satisfies the asymptotic condition

$$\Phi_j^{(\nu)}(z) \simeq \Phi_{\text{form}}^{(\nu)}(z) \qquad \text{as } z \to a_\nu, \quad z \in \Omega_{j,\nu}, \quad j = 1, \ldots, 2r_\nu + 1, \tag{2.3}$$

where $\Phi_{\text{form}}^{(\nu)}(z)$ is the formal solution at the point a_ν

$$\Phi_{\text{form}}^{(\nu)}(z) = G^{(\nu)}(z)\, e^{\Theta_\nu(z)}, \qquad G^{(\nu)}(z) = G_\nu \hat{\Phi}^{(\nu)}(z), \tag{2.4}$$

where

$$\hat{\Phi}^{(\nu)}(z) = \begin{cases} I + \sum_{k=1}^{\infty} g_{\nu,k}(z - a_\nu)^k, & \nu = 1, \ldots, n, \\ I + \sum_{k=1}^{\infty} g_{\infty,k} z^{-k}, & \nu = \infty, \end{cases}$$

and $\Theta_\nu(z)$ are diagonal matrix-valued functions

$$\Theta_\nu(z) = \begin{cases} \sum_{k=-r_\nu}^{-1} \dfrac{\Theta_{\nu,k}}{k}(z - a_\nu)^k + \Theta_{\nu,0}\ln(z - a_\nu), & \nu = 1, \ldots, n \\ -\sum_{k=1}^{r_\infty} \dfrac{\Theta_{\infty,-k}}{k} z^k - \Theta_{\infty,0}\ln z, & \nu = \infty. \end{cases}$$

We emphasize, that in (2.4) we denoted constant matrices as G_ν and matrix functions as $G^{(\nu)}$. Among the identities that determine $\Theta_\nu(z)$, $\hat{\Phi}^{(\nu)}(z)$ and $G^{(\nu)}(z)$ in terms of $A(z)$ and G_ν there is a particularly important family of relations that will be repeatedly used in what follows. Namely, the structure of the formal solution (2.4) implies that

$$A(z) - G^{(\nu)}(z)\frac{d\Theta_\nu(z)}{dz}G^{(\nu)}(z)^{-1} = \begin{cases} O(1), & \nu = 1,\ldots,n, \\ O(z^{-2}), & \nu = \infty. \end{cases} \tag{2.5}$$

The matrix $A(z)$ can thus be reconstructed by taking the sum of principal parts of Laurent series $G^{(\nu)}\Theta'_\nu G^{(\nu)^{-1}}$ at $z = a_\nu$ (plus a constant part for the point at ∞). We also can notice that according to (2.5) $\mathrm{Tr}(A(z)) = 0$ implies $\mathrm{Tr}(\Theta_\nu(z)) = 0$.

Stokes and connection matrices relate the canonical solutions $\Phi_j^{(\nu)}(z)$ in different Stokes sectors and at different singular points:

$$\Phi_{j+1}^{(\nu)} = \Phi_j^{(\nu)}S_j^{(\nu)}, \quad j = 1,\ldots,2r_\nu, \qquad \Phi_1^{(\nu)} = \Phi_1^{(\infty)}C_\nu, \quad \nu = 1,\ldots,n.$$

We can patch the Riemann-Hilbert problem from the canonical solutions $\Phi_j^{(\nu)}(z)$. Consider as an example the case with 4 singularities, $a_1 = 0$, $a_2 = 1$, $r_1 = 4$, $r_2 = 2$, $r_3 = 1$, $r_\infty = 3$.

Riemann-Hilbert Problem 2.1.

- $\Psi(z)$ *is analytic for* $z \in \mathbb{C} \setminus \Gamma$,
- $\Psi_+(z) = \Psi_-(z)J(z)$ *for* $z \in \Gamma$,
- $\Psi(z)$ *satisfies condition* (2.3) *at singular points* $0, 1, a_3, \infty$.

The contour Γ *and the associated piecewise constant jump matrices* $J(z)$ *are depicted in Figure 2.1.* z_0 *is the reference point.*

The general case can be treated in the similar way. The fact that z_0 is not singular point of $\Phi(z)$ implies the cyclic relation

$$\begin{gathered} S_1^{(\infty)}S_2^{(\infty)}\ldots S_{2r_\infty}^{(\infty)}e^{2\pi i\Theta_{\infty,0}}C_1e^{2\pi i\Theta_{1,0}}\left(S_{2r_1}^{(1)}\right)^{-1}\left(S_{2r_1-1}^{(1)}\right)^{-1}\ldots\left(S_1^{(1)}\right)^{-1}C_1^{-1} \\ \times C_2e^{2\pi i\Theta_{2,0}}\left(S_{2r_2}^{(2)}\right)^{-1}\left(S_{2r_2-1}^{(2)}\right)^{-1}\ldots\left(S_1^{(2)}\right)^{-1}C_2^{-1}\cdots \\ \times C_ne^{2\pi i\Theta_{n,0}}\left(S_{2r_n}^{(n)}\right)^{-1}\left(S_{2r_n-1}^{(n)}\right)^{-1}\ldots\left(S_1^{(n)}\right)^{-1}C_n^{-1} = I \end{gathered} \tag{2.6}$$

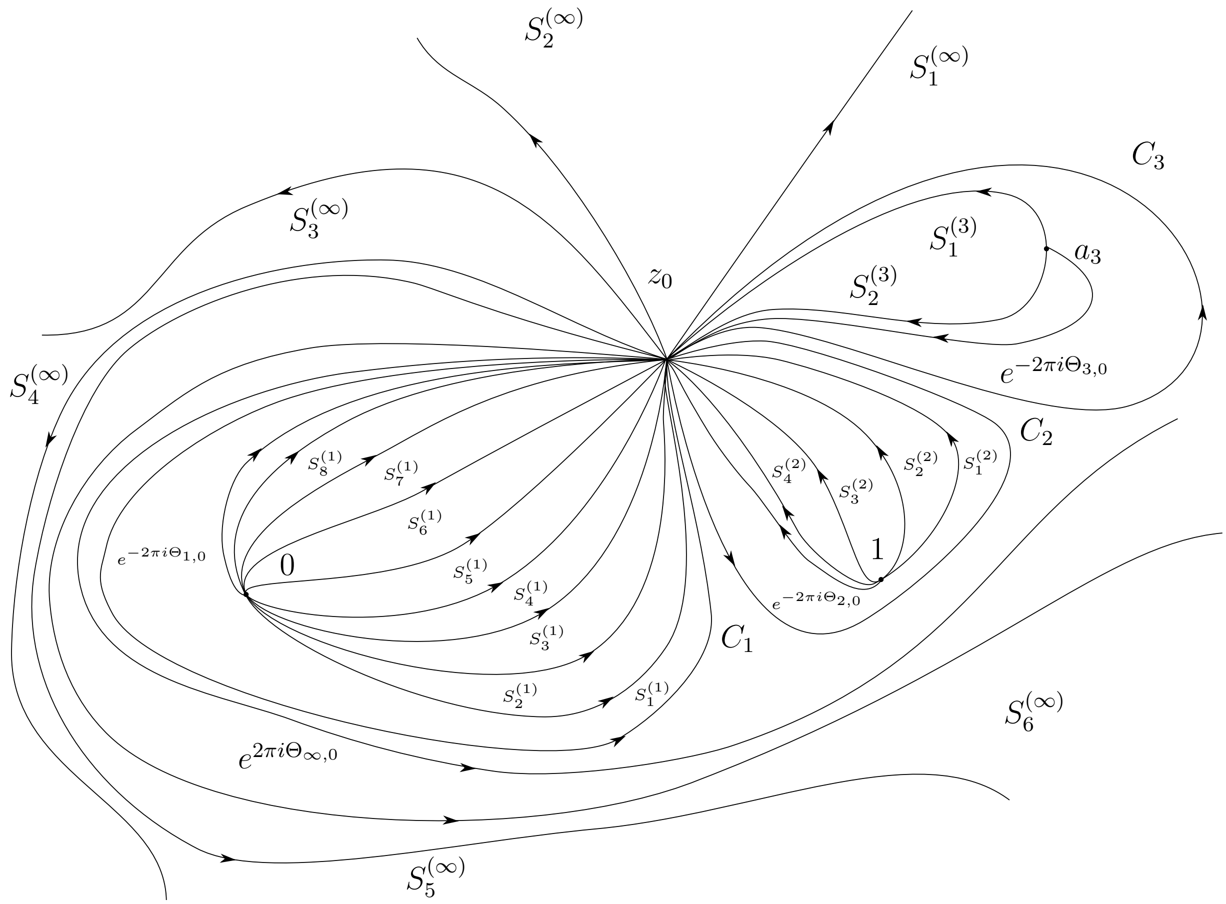

Figure 2.1. Contour Γ and jump matrices $J(z)$ for the RHP 2.1.

The identity $\mathrm{Tr}(A(z)) = 0$ implies that $\det(\Phi_j^{(\nu)}(z)) = \mathrm{const}$ by Liousville's formula. Taking the limit to singular points since $\det(G_\nu) = 1$ we get $\det(\Phi_j^{(\nu)}(z)) = 1$. Therefore

$$\det(C_\nu) = 1.$$

The standard computation [118, 49] shows that Stokes matrices $S_j^{(\nu)}$ are upper or lower triangular with ones on the diagonal.

We introduce the space $\mathcal{M}$ of monodromy data of the system (2.1) which consists of formal monodromy exponents $\Theta_{\nu,0}$, connection matrices C_ν and Stokes matrices $S_j^{(\nu)}$. More explicitly,

$$\mathcal{M} = \Big\{ S_j^{(\nu)},\, \Theta_{\nu,0} \in \mathfrak{sl}_N(\mathbb{C}),\, C_\mu \in SL_N(\mathbb{C}) : (2.6) \text{ holds},\ j = 1 \ldots 2r_\nu,$$
$$\nu = 1, \ldots, n, \infty;\ \mu = 1, \ldots, n\ \Big\} / \sim$$

The cyclic relation (2.6) puts extra N^2-1 constraints. As the result we have

$$\begin{aligned}\dim\mathcal{M} &= \left(\sum_{\nu=1}^{n} 2r_\nu + 2r_\infty\right)\frac{N(N-1)}{2} + (n+1)(N-1) + n(N^2-1) - (N^2-1)\\ &= \left(\sum_{\nu=1}^{n} r_\nu + r_\infty\right)(N^2-N) + (n+1)(N-1) + (n-1)(N^2-1).\end{aligned} \tag{2.7}$$

We can notice, that it is always even number and one can ask about symplectic form on $\mathcal{M}$, see Conjecture 2.1 below. We denote $M = (m_1, \ldots, m_{2d}) \in \mathcal{M}, \quad 2d = \dim\mathcal{M}$.

Introduce now the set of times

$$\mathcal{T} = \{a_\mu,\, \Theta_{\nu,k} \in \mathfrak{sl}_N(\mathbb{C})\,,\, k = -r_\nu, \ldots, -1;\, \nu = 1, \ldots, n, \infty;\, \mu = 1, \ldots, n\}/\sim$$

We put two constraints on this set using change of variable $z \to \alpha z + \beta$ (they are the same as for $\mathcal{A}$). We can write

$$\dim\mathcal{T} = n + \left(\sum_{\nu=1}^{n} r_\nu + r_\infty\right)(N-1) - 2 \tag{2.8}$$

All possible cases with $N = 2$ and $\dim\mathcal{T} = 1$ appear in the framework of Painlevé equations (see Chapter 6).

$$n = 0, \quad r_\infty = 3, \tag{PII}$$

$$n = 1, \quad r_1 = 1, \quad r_\infty = 1, \tag{PIII}$$

$$n = 1, \quad r_1 = 0, \quad r_\infty = 2, \tag{PIV}$$

$$n = 2, \quad r_1 = r_2 = 0, \quad r_\infty = 1, \tag{PV}$$

$$n = 3, \quad r_1 = r_2 = r_3 = r_\infty = 0, \tag{PVI}$$

We retain the notation $\vec{t} = (t_1, \ldots, t_L) \in \mathcal{T}, \quad L = \dim\mathcal{T}$.

The so-called Riemann-Hilbert correspondence states that, up to submanifolds where the inverse monodromy problem for (2.1) is not solvable, the space $\mathcal{A}$ can be identified with the product $\widetilde{\mathcal{T}} \times \mathcal{M}$, where $\widetilde{\mathcal{T}}$ denotes the universal covering of $\mathcal{T}$. We shall loosely write,

$$\mathcal{A} \simeq \widetilde{\mathcal{T}} \times \mathcal{M}.$$

Comparing (2.2), (2.7), and (2.8) we can notice $\dim\mathcal{A} = \dim\mathcal{M} + \dim\mathcal{T}$ as the confirmation of this fact.

Consider the deformation of system (2.1). We denote by $A(z) \equiv A\left(z;\vec{t};M\right)$, $G_\nu = G_\nu\left(\vec{t};M\right)$ the isomonodromic family having the same set $M \in \mathcal{M}$ of monodromy data. The isomonodromy implies that the corresponding solution $\Phi(z) \equiv \Phi\left(z,\vec{t}\right)$ satisfies an overdetermined system

$$\begin{cases} \dfrac{d\Phi}{dz} = A\left(z,\vec{t}\right)\Phi\left(z,\vec{t}\right), \\ d_{\mathcal{T}}\Phi = U\left(z,\vec{t}\right)\Phi\left(z,\vec{t}\right) \end{cases} \tag{2.9}$$

The coefficients of the matrix-valued differential form $U \equiv \sum_{k=1}^{L} U_k\left(z,\vec{t}\right)dt_k$ are rational in z. Their explicit form may be algorithmically deduced from the expression for $A(z)$. The compatibility of the system (2.9) implies the monodromy preserving deformation equation:

$$d_{\mathcal{T}}A = \frac{dU}{dz} + [U,A]. \tag{2.10}$$

Writing the second equation in (2.9) near singular points a_ν we also get

$$d_{\mathcal{T}}G_\nu = V\left(\vec{t}\right)G_\nu\left(\vec{t}\right) \tag{2.11}$$

Let us recall now the standard definition of the Jimbo-Miwa-Ueno differential [78, equation (5.1)],

$$\omega_{\mathrm{JMU}} = -\sum_{\nu=1,\ldots,n,\infty} \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\hat{\Phi}^{(\nu)}(z)^{-1}\frac{d\hat{\Phi}^{(\nu)}(z)}{dz}d_{\mathcal{T}}\Theta_\nu(z)\right). \tag{2.12}$$

It was shown in [78] that this 1-form is closed on solutions of the isomonodromy equation (2.10):

$$d_{\mathcal{T}}\omega_{\mathrm{JMU}} = 0.$$

Therefore one can integrate it and define Jimbo-Miwa-Ueno isomonodromic tau function by

$$\ln\tau(\vec{t_1},\vec{t_2},M) = \int_{\vec{t_1}}^{\vec{t_2}}\omega_{\mathrm{JMU}} \tag{2.13}$$

Besides the applications mentioned in the introduction, a remarkable property of this tau function $\tau\left(\vec{t_1},\vec{t_2};M\right)$, which was established in [96] is that it admits analytic continuation as an entire function to the whole universal covering $\widetilde{\mathcal{T}}$ of the parameter space $\mathcal{T}$. Furthermore, zeros of $\tau\left(\vec{t_1},\vec{t_2};M\right)$ as function of $\vec{t_2}$ correspond to the points in $\mathcal{T}$ where the inverse monodromy problem for (2.1) is not solvable for a given set M of monodromy data [89, 106] (or, equivalently, where a certain holomorphic vector bundle over $\hat{\mathbb{C}}$ determined by M becomes nontrivial). Hence the tau function plays a central role in the monodromy theory of systems of linear differential equations.

Consider the quotient space

$$\mathcal{A}_0 = \mathcal{A}/_{\{\mathcal{T} = \text{const}\}}. \tag{2.14}$$

It is known that system (2.10), (2.11) can be written as Hamiltonian system on space $\mathcal{A}_0$ (see [122]). In examples tau function turns out to be generating functions for the Hamiltonians or in other words

$$\omega_{\text{JMU}} = \sum_{k=1}^{L} H_k dt_k.$$

This precise statement depends on the choice of symplectic structure on $\mathcal{A}_0$ and on choice of Darboux coordinates and might not be true, as we will see in Chapter 6. We discuss the explicit formulas for Hamiltonians and symplectic structure for isomonodromic deformations in next section and in Chapter 5.

2.2 Extension of Jimbo-Miwa-Ueno differential form

To describe dependence of tau function on monodromy we would like to use natural extension of form ω_{JMU} on the whole space $\mathcal{A} \simeq \widetilde{\mathcal{T}} \times \mathcal{M}$ which would coincide with (2.12) when restricted to $\mathcal{T}$. Such extension was described in [89, 11]. It has been defined for solution $\Psi(z)$ of a general Riemann-Hilbert problem with contour Γ and jump matrix $J(z)$ as an integral

$$\omega_{\mathrm{MB}}(\partial) = \frac{1}{4\pi i} \int_{\Gamma} \mathrm{Tr}\left(\Psi_-^{-1}\Psi_-' \partial J\, J^{-1} + \Psi_+^{-1}\Psi_+' J^{-1}\partial J\right) dz. \tag{2.15}$$

The fact that in the isomonodromic setting this Malgrange-Bertola form could localize (i.e. the integral can be evaluated in terms of Ψ and its derivatives with respect to times and monodromy parameters) and become our form ω was first realized in the paper [72] by two of the authors in the context of Painlevé III (D_8). Shortly after M. Bertola pointed out how the localization should be carried out for general systems (2.1), see [72, Remark 3]. The result is 1-form $\omega \in \Lambda^1\left(\widetilde{\mathcal{T}} \times \mathcal{M}\right)$ given by

$$\omega = \sum_{\nu=1,\ldots,n,\infty} \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(A(z)\; dG^{(\nu)}(z)\; G^{(\nu)}(z)^{-1}\right), \tag{2.16}$$

where $d = d_{\mathcal{T}} + d_{\mathcal{M}}$.

The fact that restriction of (2.16) on isomonodromic times coincides with the form (2.12) was noticed by Jimbo, Miwa and Ueno themselves, cf. [78, Remark 5.2]. We provide the proof here.

Lemma 2.1 ([69]). *The restriction of the* 1*-form (2.16) on isomonodromic times coincides with form* (2.12)

$$\omega_{\mathrm{JMU}} = \sum_{k=1}^{L} \sum_{\nu=1,\ldots,n,\infty} \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(A\, \frac{dG^{(\nu)}}{dt_k} \left(G^{(\nu)}\right)^{-1}\right) dt_k. \tag{2.17}$$

Proof. First of all we can see

$$\left(\hat{\Phi}^{(\nu)}\right)^{-1} \frac{d\hat{\Phi}^{(\nu)}}{dz} = \left(G^{(\nu)}\right)^{-1} \frac{dG^{(\nu)}}{dz}.$$

Then noticing that (2.4) and (2.1) imply

$$\left(G^{(\nu)}\right)^{-1} \frac{dG^{(\nu)}}{dz} = \left(G^{(\nu)}\right)^{-1} AG^{(\nu)} - \frac{d\Theta_\nu}{dz} \tag{2.18}$$

and plugging this into the right hand side of (2.12), we have,

$$\begin{aligned}\omega_{\rm JMU} = &-\sum_{k=1}^{L}\sum_{\nu=1,\ldots,n,\infty} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(\left(G^{(\nu)}\right)^{-1} AG^{(\nu)}\frac{d\Theta_\nu}{dt_k}\right)dt_k \\ &+\sum_{k=1}^{L}\sum_{\nu=1,\ldots,n,\infty} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(\frac{d\Theta_\nu}{dz}\frac{d\Theta_\nu}{dt_k}\right)dt_k.\end{aligned} \tag{2.19}$$

The expression $\frac{d\Theta_\nu}{dz}\frac{d\Theta_\nu}{dt_k}$ has poles of order at least 2, so it does not have residues and hence the second sum in (2.19) vanishes. We also have (2.4) and (2.9) implying

$$\frac{d\Theta_\nu}{dt_k} = \left(G^{(\nu)}\right)^{-1} U_k G^{(\nu)} - \left(G^{(\nu)}\right)^{-1}\frac{dG^{(\nu)}}{dt_k}. \tag{2.20}$$

Substituting (2.20) into (2.19) we transform it to the equation

$$\omega_{\rm JMU} = -\sum_{k=1}^{L}\sum_{\nu=1,\ldots,n,\infty} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\Big(AU_k\Big)dt_k + \sum_{k=1}^{L}\sum_{\nu=1,\ldots,n,\infty} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(A\frac{dG^{(\nu)}}{dt_k}\left(G^{(\nu)}\right)^{-1}\right)dt_k.$$

The function $\operatorname{Tr}\Big(AU_k\Big)$ is rational, therefore the sum of its residues is zero. So we get (2.17).

□

The expression (2.16) has the important property.

Lemma 2.2 ([69])**.** *The form $d\omega$ has no cross terms of the kind $dt_k \wedge dm_j$, $k = 1, \ldots, L, \quad j = 1, \ldots, 2d$.*

Proof. Denote

$$I = \sum_{\nu,k,j} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(\frac{\partial}{\partial m_j}\left(A\frac{dG^{(\nu)}}{dt_k}\left(G^{(\nu)}\right)^{-1}\right) - \frac{d}{dt_k}\left(A\frac{\partial G^{(\nu)}}{\partial m_j}\left(G^{(\nu)}\right)^{-1}\right)\right).$$

We have

$$\begin{aligned}I = &\sum_{\nu,k,j} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(\frac{\partial A}{\partial m_j}\frac{dG^{(\nu)}}{dt_k}\left(G^{(\nu)}\right)^{-1} - A\frac{dG^{(\nu)}}{dt_k}\left(G^{(\nu)}\right)^{-1}\frac{\partial G^{(\nu)}}{\partial m_j}\left(G^{(\nu)}\right)^{-1}\right) \\ &+\sum_{\nu,k,j} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(A\frac{\partial G^{(\nu)}}{\partial m_j}\left(G^{(\nu)}\right)^{-1}\frac{dG^{(\nu)}}{dt_k}\left(G^{(\nu)}\right)^{-1} - \frac{dA}{dt_k}\frac{\partial G^{(\nu)}}{\partial m_j}\left(G^{(\nu)}\right)^{-1}\right).\end{aligned}$$

We use the formula (2.20) to get rid of $\frac{dG^{(\nu)}}{dt_k}$ and equation (2.10) to replace $\frac{dA}{dt_k}$. The expression $\operatorname{Tr}\left(\frac{\partial A}{\partial m_j}U_k\right)$ is rational function and some of its residues is zero. After some cancellations we have

$$I = \sum_{\nu,k,j} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(-\frac{\partial A}{\partial m_j}G^{(\nu)}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1} + AG^{(\nu)}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\frac{\partial G^{(\nu)}}{\partial m_j}\left(G^{(\nu)}\right)^{-1}\right)$$

$$-\sum_{\nu,k,j}\mathrm{res}_{z=a_\nu}\,\mathrm{Tr}\left(\frac{dU_k}{dz}\frac{\partial G^{(\nu)}}{\partial m_j}\left(G^{(\nu)}\right)^{-1}+A\frac{\partial G^{(\nu)}}{\partial m_j}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\right).$$

We replace U_k using again formula (2.20).

$$\begin{aligned}I=&\sum_{\nu,k,j}\mathrm{res}_{z=a_\nu}\,\mathrm{Tr}\left(-\frac{\partial A}{\partial m_j}G^{(\nu)}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}+AG^{(\nu)}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\frac{\partial G^{(\nu)}}{\partial m_j}\left(G^{(\nu)}\right)^{-1}\right)\\&-\sum_{\nu,k,j}\mathrm{res}_{z=a_\nu}\,\mathrm{Tr}\left(\frac{d}{dz}\left(G^{(\nu)}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\right)\frac{\partial G^{(\nu)}}{\partial m_j}\left(G^{(\nu)}\right)^{-1}\right)\qquad(2.21)\\&-\sum_{\nu,k,j}\mathrm{res}_{z=a_\nu}\,\mathrm{Tr}\left(\frac{d}{dz}\left(\frac{dG^{(\nu)}}{dt_k}\left(G^{(\nu)}\right)^{-1}\right)\frac{\partial G^{(\nu)}}{\partial m_j}\left(G^{(\nu)}\right)^{-1}+A\frac{\partial G^{(\nu)}}{\partial m_j}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\right).\end{aligned}$$

The fourth term is regular near $z=a_\nu$, therefore its residue is zero. Now we notice the important fact that the residue of the derivative with respect to z of formal series is zero. Therefore we can "integrate by parts", moving the derivative from one term to another

$$0=\mathrm{res}_{z=a_\nu}(fg)'=\mathrm{res}_{z=a_\nu}f'g+\mathrm{res}_{z=a_\nu}fg'.\qquad(2.22)$$

We do that with the third term in (2.21).

$$\begin{aligned}I=&\sum_{\nu,k,j}\mathrm{res}_{z=a_\nu}\,\mathrm{Tr}\left(-\frac{\partial A}{\partial m_j}G^{(\nu)}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}+AG^{(\nu)}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\frac{dG^{(\nu)}}{\partial m_j}\left(G^{(\nu)}\right)^{-1}\right)\\&+\sum_{\nu,k,j}\mathrm{res}_{z=a_\nu}\,\mathrm{Tr}\left(-A\frac{\partial G^{(\nu)}}{\partial m_j}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}+G^{(\nu)}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\frac{\partial}{\partial m_j}\left(\frac{dG^{(\nu)}}{dz}\right)\left(G^{(\nu)}\right)^{-1}\right)\\&-\sum_{\nu,k,j}\mathrm{res}_{z=a_\nu}\,\mathrm{Tr}\left(\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\frac{\partial G^{(\nu)}}{\partial m_j}\left(G^{(\nu)}\right)^{-1}\frac{\partial G^{(\nu)}}{\partial z}\right).\end{aligned}$$

Using (2.18) to replace $\dfrac{\partial G^{(\nu)}}{\partial z}$, we have

$$\begin{aligned}I=&\sum_{\nu,k,j}\mathrm{res}_{z=a_\nu}\,\mathrm{Tr}\left(-\frac{\partial A}{\partial m_j}G^{(\nu)}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}-A\frac{\partial G^{(\nu)}}{\partial m_j}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\right)\\&+\sum_{\nu,k,j}\mathrm{res}_{z=a_\nu}\,\mathrm{Tr}\left(G^{(\nu)}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\frac{\partial}{\partial m_j}\left(\frac{dG^{(\nu)}}{dz}\right)\left(G^{(\nu)}\right)^{-1}+\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\frac{\partial G^{(\nu)}}{\partial m_j}\frac{\partial\Theta_\nu}{\partial z}\right)\end{aligned}$$

Finally using (2.18) one more time we get

$$\frac{\partial}{\partial m_j}\left(\frac{dG^{(\nu)}}{dz}\right)=\frac{\partial A}{\partial m_j}G^{(\nu)}+A\frac{\partial G^{(\nu)}}{\partial m_j}-\frac{\partial G^{(\nu)}}{\partial m_j}\frac{\partial\Theta_\nu}{\partial z}-G^{(\nu)}\frac{\partial}{\partial m_j}\left(\frac{d\Theta_\nu}{dz}\right)$$

Since $\mathrm{Tr}\left(\dfrac{d\Theta_\nu}{dt_k}\dfrac{\partial}{\partial m_j}\left(\dfrac{d\Theta_\nu}{dz}\right)\right)$ has pole at least of order 2, we get $I=0$. □

We have the following corollary of Lemma 2.2.

Corollary 2.1 ([69])**.** *Form $d\omega$ is closed form on $\mathcal{M}$ independent on $\mathcal{T}$.*

Proof. We notice that

$$0 = d(d\omega) = d_{\mathcal{T}}(d\omega) + d_{\mathcal{M}}(d\omega).$$

Since ω_{JMU} is closed form and $d\omega$ has no cross terms by Lemma 2.2, form $d\omega$ contains only differentials with respect to m_j. Therefore we have separately

$$d_{\mathcal{T}}(d\omega) = 0, \quad d_{\mathcal{M}}(d\omega) = 0.$$

Therefore the claim follows. □

Form $d\omega$ can be computed in terms of monodromy data using the relation with form (2.15) and results of [10]. We arrive to the following conjecture.

Conjecture 2.1 ([73, 69])**.** *Form $d\omega$ is nondegenerate form on $\mathcal{M}$.*

If it is true, then form $d\omega$ gives symplectic structure on $\mathcal{M}$. The examples, when Conjecture 2.1 holds are provided in Remark 4.2, Remark 4.4.

Lemma 2.2 also plays a crucial role in rigorous solution of connection problem. Indeed, a direct corollary of Lemma 2.1 is the following integral formula for the tau function (2.13),

$$\ln \tau(\vec{t_1}, \vec{t_2}, M) = \int_{\vec{t_1}}^{\vec{t_2}} \sum_{k=1}^{L} \sum_{\nu=1,\dots,n,\infty} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(A \frac{dG^{(\nu)}}{dt_k} \left(G^{(\nu)}\right)^{-1} \right) dt_k. \tag{2.23}$$

A key issue in the determining of the monodromy dependence of the tau function is the possibility of the effective evaluation of the derivative of the integral (2.23) with respect to the monodromy parameters m_j. Lemma 2.2 implies that

$$\frac{\partial}{\partial m_j} \sum_{\nu} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(A \frac{dG^{(\nu)}}{dt_k} \left(G^{(\nu)}\right)^{-1} \right) = \frac{d}{dt_k} \sum_{\nu} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(A \frac{\partial G^{(\nu)}}{\partial m_j} \left(G^{(\nu)}\right)^{-1} \right).$$

Therefore,

$$\begin{aligned} \frac{\partial \ln \tau}{\partial m_j} &= \int_{\vec{t_1}}^{\vec{t_2}} \sum_{k=1}^{L} \frac{\partial}{\partial m_j} \sum_{\nu=1,\dots,n,\infty} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(A \frac{dG^{(\nu)}}{dt_k} \left(G^{(\nu)}\right)^{-1} \right) dt_k \\ &= \sum_{k=1}^{L} \int_{\vec{t_1}}^{\vec{t_2}} \frac{d}{dt_k} \sum_{\nu=1,\dots,n,\infty} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(A \frac{\partial G^{(\nu)}}{\partial m_j} \left(G^{(\nu)}\right)^{-1} \right) dt_k \\ &= \sum_{\nu=1,\dots,n,\infty} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(A \frac{\partial G^{(\nu)}}{\partial m_j} \left(G^{(\nu)}\right)^{-1} \right)\Bigg|_{\vec{t_1}}^{\vec{t_2}}. \end{aligned} \tag{2.24}$$

In other words, we conclude that in addition to the differential relation

$$d_{\mathcal{T}} \ln \tau = \sum_{\nu=1,\ldots,n,\infty} \mathrm{res}_{z=a_\nu} \mathrm{Tr} \left(G^{(\nu)}(z)^{-1} A(z)\, d_{\mathcal{T}} G^{(\nu)}(z) \right),$$

the tau function satisfies the differential relation,

$$d_{\mathcal{M}} \ln \tau = \sum_{\nu=1,\ldots,n,\infty} \mathrm{res}_{z=a_\nu} \mathrm{Tr} \left(G^{(\nu)}(z)^{-1} A(z)\, d_{\mathcal{M}} G^{(\nu)}(z) \right).$$

These two differential identities allow to evaluate the asymptotic connection formulae up to the numerical constants. Integrating (2.24) with respect to monodromy we can sum up the above discussion in the following way

Theorem 2.1 ([73])**.** *The tau function* (2.13) *can be evaluated alternatively as*

$$\ln \tau(\vec{t_1}, \vec{t_2}, M) = \ln \tau(\vec{t_1}, \vec{t_2}, M_0) + \int_{M_0}^{M} \sum_{\nu,j} \mathrm{res}_{a_\nu} \mathrm{Tr} \left(A \frac{\partial G^{(\nu)}}{\partial m_j} \left(G^{(\nu)}\right)^{-1} \right) \Bigg|_{\vec{t_1}}^{\vec{t_2}} dm_j. \tag{2.25}$$

The arguments which led to the representation (2.24) for the logarithmic derivative of the tau function with respect to m_j are reminiscent to the variational equations for the classical action. Let us assume that we can identify the classical Darboux coordinates [1],

$$\vec{p} = (p_1, \ldots, p_d), \quad \vec{q} = (q_1, \ldots, q_d)$$

on the space $\mathcal{A}_0$ (given by (2.14)) so that the isomonodromic deformation equations (2.10), (2.11) can be written as the commuting system of Hamiltonian dynamical equations,

$$\frac{dq_j}{dt_k} = \frac{\partial H_k}{\partial p_j}, \quad \frac{dp_j}{dt_k} = -\frac{\partial H_k}{\partial q_j}, \quad k = 1, \ldots, L, \quad j = 1, \ldots, d \tag{2.26}$$

The compatibility of the system (2.26) means (see, e.g., [2]) that all

$$c_{kl} := 2\{H_k, H_l\} + \frac{\partial H_k}{\partial t_l} - \frac{\partial H_l}{\partial t_k}, \quad k, l = 1, \ldots, L$$

are the Casimir functions[2] (maybe depending on the times t_k), that is

$$\{c_{kl}, f\} = 0, \quad \text{for any function} \quad f = f(\vec{q}, \vec{p}). \tag{2.27}$$

[1]The Darboux coordinates on the phase spaces $\mathcal{A}_0$ corresponding to Painlevé equations are introduced in [63], [2];

[2]Warning: here,

$$\frac{\partial H_k}{\partial t_l} = \frac{\partial}{\partial t_l} \left(H_k(\vec{p}, \vec{q}, \vec{t})\big|_{\vec{p},\vec{q} \equiv \mathrm{const}} \right).$$

The *classical action differential* can be defined as the differential form on $\widetilde{\mathcal{T}} \times \mathcal{M}$,

$$\omega_{\text{cla}} = \sum_j p_j dq_j - \sum_k H_k dt_k \equiv \sum_k \left(\sum_j p_j \frac{dq_j}{dt_k} - H_k \right) dt_k + \sum_k \left(\sum_j p_j \frac{\partial q_j}{\partial m_k} \right) dm_k$$

We shall assume that

$$\{H_k, H_l\} + \frac{\partial H_k}{\partial t_l} - \frac{\partial H_l}{\partial t_k} = 0 \quad k, l = 1, \ldots, L. \tag{2.28}$$

Using (2.27), (2.28) it is easy to check that it is closed on the trajectories of the dynamical system (2.26), i.e.,

$$d_{\mathcal{T}} \left(\omega_{\text{cla}} \big|_{M \equiv \text{const}} \right) = 0.$$

Note that in those cases when the logarithm of the tau function is the generating function for the Hamiltonians H_k, the Jimbo-Miwa-Ueno differential form is

$$\omega_{\text{JMU}} = \sum_k H_k dt_k,$$

so that the integral (2.23) is the *truncated* action integral,

$$\ln \tau = \int_{\vec{t_1}}^{\vec{t_2}} \sum_k H_k dt_k.$$

Suppose that instead of this integral we need to study the complete action, i.e. the integral,

$$S \equiv S(\vec{t_1}, \vec{t_2}, M) = \int_{\vec{t_1}}^{\vec{t_2}} \omega_{\text{cla}}(M) \equiv \int_{\vec{t_1}}^{\vec{t_2}} \sum_k \left(\sum_j p_j \frac{dq_j}{dt_k} - H_k \right) dt_k.$$

Then, the usual variational calculus arguments show that in any m_j–derivative of S the integral terms would disappear. In fact, assume that Hamiltonians H_k do not depend explicitly on m_j. We have,

$$\begin{aligned} \frac{\partial S}{\partial m_{j_0}} &= \int_{\vec{t_1}}^{\vec{t_2}} \sum_k \left(\sum_j \frac{\partial p_j}{\partial m_{j_0}} \frac{dq_j}{dt_k} + p_j \frac{\partial}{\partial m_{j_0}} \left(\frac{dq_j}{dt_k} \right) - \frac{\partial H_k}{\partial p_j} \frac{\partial p_j}{\partial m_{j_0}} - \frac{\partial H_k}{\partial q_j} \frac{\partial q_j}{\partial m_{j_0}} \right) dt_k \\ &= \sum_j p_j \frac{\partial q_j}{\partial m_{j_0}} \Bigg|_{\vec{t_1}}^{\vec{t_2}} + \int_{\vec{t_1}}^{\vec{t_2}} \sum_k \left(\sum_j \frac{\partial p_j}{\partial m_{j_0}} \frac{dq_j}{dt_k} - \frac{\partial q_j}{\partial m_{j_0}} \frac{dp_j}{dt_k} - \frac{\partial H_k}{\partial p_j} \frac{\partial p_j}{\partial m_{j_0}} - \frac{\partial H_k}{\partial q_j} \frac{\partial q_j}{\partial m_{j_0}} \right) dt_k \quad (2.29) \\ &= \sum_j p_j \frac{\partial q_j}{\partial m_{j_0}} \Bigg|_{\vec{t_1}}^{\vec{t_2}} \end{aligned}$$

and the integral term vanishes because of the equations of motion (2.26). Comparison (2.24) and (2.29) makes one to suspect some deep connection between the tau function and the classical action. Indeed, taking the full exterior derivation of $\omega_{\text{cla}} \equiv \omega_{\text{cla}}(\vec{t}, M)$, one obtains,

$$d\omega_{\text{cla}} = \sum_j d_{\mathcal{M}} p_j \wedge d_{\mathcal{M}} q_j. \tag{2.30}$$

Note, that both, the form $d\omega$ and the form $d\omega_{\text{cla}}$ are the closed 2– forms on $\mathcal{M}$ and they do not depend on the times $\mathcal{T}$. Therefore we can expect that the two 1-forms, ω and ω_{cla}, coincide up to the total differential. We formulate it as the conjecture.

Conjecture 2.2 ([73]). *There exists a rational function $G(\vec{p}, \vec{q}, \vec{t})$ of $\vec{p}, \vec{q}, \vec{t}$ such that,*

$$\omega = \omega_{cla} + dG(\vec{p}, \vec{q}, \vec{t}). \tag{2.31}$$

Moreover, the function $G(\vec{p}, \vec{q}, \vec{t})$ is explicitly computable.

Identity (2.30) means that Conjecture 2.2 implies Conjecture 2.1.

The statement of Conjecture 2.2 has been proven to be true in the case of the Sine-Gordon reduction of Painlevé III equation [72], in the case of the (homogenous) Painlevé II equation [69], and in the case of the Painlevé I equation [87]. In the Chapter 6 of this paper we demonstrate the validity of this conjecture for the rest of the Painlevé equations following [73].

Restricting (2.31) to the isomonodromic family $M \equiv \text{const}$, one arrives to the identity

$$\begin{aligned} \sum_k \frac{d\ln\tau}{dt_k} dt_k = \sum_k \left(\sum_j p_j(\vec{t}, M) \frac{dq_j(\vec{t}, M)}{dt_k} - H_k\Big(\vec{p}(\vec{t}, M), \vec{q}(\vec{t}, M), \vec{t}\Big) \right) dt_k \\ + \sum_k \frac{d}{dt_k} G\Big(\vec{p}(\vec{t}, M), \vec{q}(\vec{t}, M), \vec{t}\Big) dt_k. \end{aligned} \tag{2.32}$$

and hence,

$$\ln\tau(\vec{t_1}, \vec{t_2}, M) = S(\vec{t_1}, \vec{t_2}, M) + G\Big(\vec{p}(\vec{t}, M), \vec{q}(\vec{t}, M), \vec{t}\Big)\Big|_{\vec{t_1}}^{\vec{t_2}}. \tag{2.33}$$

This, in turn, would produce, taking into account (2.29), the following, alternative to (2.24), formula for the m_j – derivative of $\ln\tau$,

$$\frac{\partial \ln\tau}{\partial m_{j_0}} = \sum_j p_j \frac{\partial q_j}{\partial m_{j_0}}\Big|_{\vec{t_1}}^{\vec{t_2}} + \frac{\partial G}{\partial m_{j_0}}\Big|_{\vec{t_1}}^{\vec{t_2}}. \tag{2.34}$$

This version of the variational logarithmic derivatives of the tau functions turns out even more efficient then (2.24) in the concrete examples related to the “constant problem”. Indeed, the particular cases of (2.34) have been used in [22] in evaluation of the constant terms in the asymptotics of the several basic distribution functions of random matrix theory expressible in terms of the Painlevé transcendents. We provide the derivation of identities similar to (2.33) for Painlevé equations in Chapter 3 and use them in Section 4.1 and Section 4.2 for solution of connection problems.

3. PROPERTY OF QUASIHOMOGENEOUS HAMILTONIANS

For the Painlevé equations identities (2.33), (2.34) can be obtained using the general property of their Hamiltonians. We start with the definition.

Definition 3.1. *The function* $f : \mathbb{C}^n \to \mathbb{C}$ *is called quasihomogeneous if* $\exists w, w_1, w_2, \dots w_n \in \mathbb{Z}$ *such that* $\forall \lambda \in \mathbb{R},\ \lambda > 0$

$$f(\lambda^{w_1} x_1, \lambda^{w_2} x_2, \dots, \lambda^{w_n} x_n) = \lambda^w f(x_1, x_2, \dots, x_n). \tag{3.1}$$

Taking the derivative of (3.1) with respect to λ at $\lambda = 1$ we get

$$w_1 x_1 \frac{\partial f}{\partial x_1} + w_2 x_2 \frac{\partial f}{\partial x_2} + \dots + w_n x_n \frac{\partial f}{\partial x_n} = wf. \tag{3.2}$$

The Hamiltonians for Painlevé equations are listed in [104, 80]. We write the version of them, corresponding to equations of motions PI(F) – PVI(F)

$$H = \frac{p^2}{2} - 2q^3 - tq, \tag{PI}$$

$$H = \frac{p^2}{2} - \frac{q^4}{2} - \frac{q^2 t}{2} - q\alpha, \tag{PII}$$

$$H = \frac{p^2}{2} - \alpha e^{t+q} + \beta e^{t-q} - \frac{\gamma}{2} e^{2t+2q} + \frac{\delta}{2} e^{2t-2q}, \tag{PIII}$$

$$H = \frac{p^2}{2} - \frac{q^6}{8} - \frac{tq^4}{2} - \frac{q^2}{2}(t^2 - \alpha) + \frac{\beta}{4q^2}, \tag{PIV}$$

$$H = \frac{p^2}{2} - \frac{\alpha}{\sinh^2(\frac{q}{2})} - \frac{\beta}{\cosh^2(\frac{q}{2})} + \frac{\gamma}{2} e^t \cosh(q) + \frac{\delta}{4} e^{2t} \cosh(2q), \tag{PV}$$

$$H = \frac{p^2}{2} - \frac{K^2}{\pi^2} \left[\alpha k^2 \mathrm{sn}^2(2qK, k) - \frac{\beta}{\mathrm{sn}^2(2qK, k)} + \frac{\gamma(1-k^2)}{\mathrm{cn}^2(2qK, k)} \right.$$
$$\left. + \left(\delta - \frac{1}{2}\right) \frac{(1-k^2)}{\mathrm{dn}^2(2qK, k)} \right], \tag{PVI}$$

We can notice the following properties

$$H(\lambda^3 p, \lambda^2 q, \lambda^4 t) = \lambda^6 H(p,q,t), \tag{PI}$$

$$H(\lambda^2 p, \lambda q, \lambda^2 t, \lambda^3 \alpha) = \lambda^4 H(p,q,t,\alpha), \tag{PII}$$

$$H(\lambda p, q, t + \ln\lambda, \lambda\alpha, \lambda\beta) = \lambda^2 H(p,q,t,\alpha,\beta), \tag{PIII(D6)}$$

$$H(\lambda^2 p, q - \ln\lambda, t + 3\ln\lambda, \lambda^2\alpha) = \lambda^4 H(p,q,t,\alpha), \tag{PIII(D7)}$$

$$H(\lambda p, q, t + 2\ln\lambda) = \lambda^2 H(p,q,t), \tag{PIII(D8)}$$

$$H(\lambda^3 p, \lambda q, \lambda^2 t, \lambda^4\alpha, \lambda^8\beta) = \lambda^6 H(p,q,t,\alpha,\beta), \tag{PIV}$$

$$H(\lambda p, q, t + \ln\lambda, \lambda^2\alpha, \lambda^2\beta, \lambda\gamma) = \lambda^2 H(p,q,t,\alpha,\beta,\gamma), \tag{PV}$$

$$H\left(\lambda p, q, t, \lambda^2\alpha, \lambda^2\beta, \lambda^2\gamma, \frac{1}{2} + \lambda^2\left(\delta - \frac{1}{2}\right)\right) = \lambda^2 H(p,q,t,\alpha,\beta,\gamma,\delta), \tag{PVI}$$

They mean that up to change of variables, the Hamiltonians are quasihomogeneous functions. The consequence of this fact is that Painlevé equations can be interpreted as dynamical system on weighted projective spaces [26, 27].

We use these properties to derive identities (3.2)

$$3p\frac{\partial H}{\partial p} + 2q\frac{\partial H}{\partial q} + 4t\frac{\partial H}{\partial t} = 6H, \tag{PI}$$

$$2p\frac{\partial H}{\partial p} + q\frac{\partial H}{\partial q} + 2t\frac{\partial H}{\partial t} + 3\alpha\frac{\partial H}{\partial \alpha} = 4H, \tag{PII}$$

$$p\frac{\partial H}{\partial p} + \frac{\partial H}{\partial t} + \alpha\frac{\partial H}{\partial \alpha} + \beta\frac{\partial H}{\partial \beta} = 2H, \tag{PIII(D6)}$$

$$2p\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q} + 3\frac{\partial H}{\partial t} + 2\alpha\frac{\partial H}{\partial \alpha} = 4H, \tag{PIII(D7)}$$

$$p\frac{\partial H}{\partial p} + 2\frac{\partial H}{\partial t} = 2H, \tag{PIII(D8)}$$

$$3p\frac{\partial H}{\partial p} + q\frac{\partial H}{\partial q} + 2t\frac{\partial H}{\partial t} + 4\alpha\frac{\partial H}{\partial \alpha} + 8\beta\frac{\partial H}{\partial \beta} = 6H, \tag{PIV}$$

$$p\frac{\partial H}{\partial p} + \frac{\partial H}{\partial t} + 2\alpha\frac{\partial H}{\partial \alpha} + 2\beta\frac{\partial H}{\partial \beta} + \gamma\frac{\partial H}{\partial \gamma} = 2H, \tag{PV}$$

$$p\frac{\partial H}{\partial p} + 2\alpha\frac{\partial H}{\partial \alpha} + 2\beta\frac{\partial H}{\partial \beta} + 2\gamma\frac{\partial H}{\partial \gamma} + (2\delta - 1)\frac{\partial H}{\partial \delta} = 2H, \tag{PVI}$$

Using the equations of motion we can rewrite them as

$$5H = 5\left(p\frac{dq}{dt} - H\right) + \frac{d}{dt}(4tH - 2pq), \tag{PI}$$

$$3H = 3\left(p\frac{dq}{dt} - H\right) + \frac{d}{dt}(2tH - pq) + 3\alpha\frac{\partial H}{\partial \alpha}, \tag{PII}$$

$$H = \left(p\frac{dq}{dt} - H\right) + \frac{dH}{dt} + \alpha\frac{\partial H}{\partial \alpha} + \beta\frac{\partial H}{\partial \beta}, \tag{PIII(D6)}$$

$$H = \left(p\frac{dq}{dt} - H\right) + \frac{3}{2}\frac{dH}{dt} + \frac{1}{2}\frac{dp}{dt} + \alpha\frac{\partial H}{\partial \alpha}, \tag{PIII(D7)}$$

$$H = \left(p\frac{dq}{dt} - H\right) + 2\frac{dH}{dt}, \tag{PIII(D8)}$$

$$4H = 4\left(p\frac{dq}{dt} - H\right) + \frac{d}{dt}(2tH - pq) + 4\alpha\frac{\partial H}{\partial \alpha} + 8\beta\frac{\partial H}{\partial \beta}, \tag{PIV}$$

$$H = \left(p\frac{dq}{dt} - H\right) + \frac{dH}{dt} + 2\alpha\frac{\partial H}{\partial \alpha} + 2\beta\frac{\partial H}{\partial \beta} + \gamma\frac{\partial H}{\partial \gamma}, \tag{PV}$$

$$H = \left(p\frac{dq}{dt} - H\right) + 2\alpha\frac{\partial H}{\partial \alpha} + 2\beta\frac{\partial H}{\partial \beta} + 2\gamma\frac{\partial H}{\partial \gamma} + (2\delta - 1)\frac{\partial H}{\partial \delta}. \tag{PVI}$$

We introduce the classical action by the formula

$$S(t_1, t_2) = \int_{t_1}^{t_2} \left(p\frac{dq}{dt} - H\right) dt \tag{3.3}$$

and we remind that the tau function is given by

$$\ln \tau(t_1, t_2) = \int_{t_1}^{t_2} H dt.$$

We can write the following formula for derivative of (3.3) with respect to parameter ρ generalizing (2.29).

$$\begin{aligned}\frac{\partial S}{\partial \rho} &= \int_{t_1}^{t_2} \left(\frac{\partial p}{\partial \rho}\frac{dq}{dt} + p\frac{d}{dt}\left(\frac{\partial q}{\partial \rho}\right) - \frac{\partial H}{\partial p}\frac{\partial p}{\partial \rho} - \frac{\partial H}{\partial q}\frac{\partial q}{\partial \rho} - \frac{\partial H}{\partial \rho}\right) dt \\ &= p\frac{\partial q}{\partial \rho}\bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial p}{\partial \rho}\frac{dq}{dt} - \frac{\partial q}{\partial \rho}\frac{dp}{dt} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial \rho} - \frac{\partial H}{\partial q}\frac{\partial q}{\partial \rho} - \frac{\partial H}{\partial \rho}\right) dt \\ &= p\frac{\partial q}{\partial \rho}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial H}{\partial \rho} dt\end{aligned} \tag{3.4}$$

For the second equality we integrated by parts and for the third equality we used equations of motion. Assume that solutions of Painlevé equation with fixed parameters $\alpha, \beta, \gamma, \delta$ are parameterized by m_1 and m_2. Then

$$\frac{\partial H}{\partial m_j} = 0.$$

And we can write the alternative representation of classical action

$$S(t_1, t_2, M) = S(t_1, t_2, M_0) + \int_{M_0}^{M} \sum_{j=1}^{2} p \frac{\partial q}{\partial m_j}\Bigg|_{t_1}^{t_2} dm_j. \tag{3.5}$$

Using this formula and (3.4) we also get

$$\int_{t_1}^{t_2} \frac{\partial H}{\partial \rho} dt = p \frac{\partial q}{\partial \rho}\Bigg|_{t_1}^{t_2} - \frac{\partial}{\partial \rho} S(t_1, t_2, M_0) - \frac{\partial}{\partial \rho} \int_{M_0}^{M} \sum_{j=1}^{2} p \frac{\partial q}{\partial m_j}\Bigg|_{t_1}^{t_2} dm_j. \tag{3.6}$$

The formula of sort (3.6) was obtained and used first time in [23] for PIII(D6) equation. Integrating identities for Hamiltonians and using (3.5), (3.6) with $\rho = \alpha, \beta, \gamma, \delta$ we get

Theorem 3.1. *Painlevé tau functions admit the following representations similar to* (2.25)

$$\ln\tau(t_1,t_2,M) = \frac{1}{5}(4tH-2pq)\Big|_{t_1}^{t_2} + S(t_1,t_2,M_0) + \int\limits_{M_0}^{M}\sum_{j=1}^{2} p\frac{\partial q}{\partial m_j}\Big|_{t_1}^{t_2} dm_j, \tag{PI}$$

$$\ln\tau(t_1,t_2,M) = \left(\frac{2}{3}tH - \frac{1}{3}pq + \alpha p\frac{\partial q}{\partial\alpha}\right)\Big|_{t_1}^{t_2} + \left(1-\alpha\frac{\partial}{\partial\alpha}\right)\left(S(t_1,t_2,M_0) + \int\limits_{M_0}^{M}\sum_{j=1}^{2} p\frac{\partial q}{\partial m_j}\Big|_{t_1}^{t_2} dm_j\right), \tag{PII}$$

$$\ln\tau(t_1,t_2,M) = \left(H + \alpha p\frac{\partial q}{\partial\alpha} + \beta p\frac{\partial q}{\partial\beta}\right)\Big|_{t_1}^{t_2} + \left(1-\alpha\frac{\partial}{\partial\alpha} - \beta\frac{\partial}{\partial\beta}\right)\left(S(t_1,t_2,M_0) + \int\limits_{M_0}^{M}\sum_{j=1}^{2} p\frac{\partial q}{\partial m_j}\Big|_{t_1}^{t_2} dm_j\right), \tag{PIII(D6)}$$

$$\ln\tau(t_1,t_2,M) = \left(\frac{3H}{2} + \frac{p}{2} + \alpha p\frac{\partial q}{\partial\alpha}\right)\Big|_{t_1}^{t_2} + \left(1-\alpha\frac{\partial}{\partial\alpha}\right)\left(S(t_1,t_2,M_0) + \int\limits_{M_0}^{M}\sum_{j=1}^{2} p\frac{\partial q}{\partial m_j}\Big|_{t_1}^{t_2} dm_j\right), \tag{PIII(D7)}$$

$$\ln\tau(t_1,t_2,M) = 2H|_{t_1}^{t_2} + S(t_1,t_2,M_0) + \int\limits_{M_0}^{M}\sum_{j=1}^{2} p\frac{\partial q}{\partial m_j}\Big|_{t_1}^{t_2} dm_j, \tag{PIII(D8)}$$

$$\ln\tau(t_1,t_2,M) = \left(\frac{1}{2}tH - \frac{1}{4}pq + \alpha p\frac{\partial q}{\partial\alpha} + 2\beta p\frac{\partial q}{\partial\beta}\right)\Big|_{t_1}^{t_2} + \left(1-\alpha\frac{\partial}{\partial\alpha} - 2\beta\frac{\partial}{\partial\beta}\right)\left(S(t_1,t_2,M_0) + \int\limits_{M_0}^{M}\sum_{j=1}^{2} p\frac{\partial q}{\partial m_j}\Big|_{t_1}^{t_2} dm_j\right), \tag{PIV}$$

$$\ln\tau(t_1,t_2,M) = \left(H + 2\alpha p\frac{\partial q}{\partial\alpha} + 2\beta p\frac{\partial q}{\partial\beta} + \gamma p\frac{\partial q}{\partial\gamma}\right)\Big|_{t_1}^{t_2} + \left(1-2\alpha\frac{\partial}{\partial\alpha} - 2\beta\frac{\partial}{\partial\beta} - \gamma\frac{\partial}{\partial\gamma}\right)\left(S(t_1,t_2,M_0) + \int\limits_{M_0}^{M}\sum_{j=1}^{2} p\frac{\partial q}{\partial m_j}\Big|_{t_1}^{t_2} dm_j\right), \tag{PV}$$

$$\ln\tau(t_1,t_2,M) = \left(2\alpha p\frac{\partial q}{\partial\alpha} + 2\beta p\frac{\partial q}{\partial\beta} + 2\gamma p\frac{\partial q}{\partial\gamma} + (2\delta-1)p\frac{\partial q}{\partial\delta}\right)\Big|_{t_1}^{t_2} + \left(1-2\alpha\frac{\partial}{\partial\alpha} - 2\beta\frac{\partial}{\partial\beta} - 2\gamma\frac{\partial}{\partial\gamma} - (2\delta-1)\frac{\partial}{\partial\delta}\right)\left(S(t_1,t_2,M_0) + \int\limits_{M_0}^{M}\sum_{j=1}^{2} p\frac{\partial q}{\partial m_j}\Big|_{t_1}^{t_2} dm_j\right), \tag{PVI}$$

4. CONNECTION PROBLEM FOR PAINLEVÉ TAU FUNCTIONS

4.1 PIII(D8)(F) equation

Consider the equation PIII(D8)(F) with $\alpha = -\frac{1}{8}, \quad \beta = \frac{1}{8}, \quad \gamma = \delta = 0$

$$\frac{d^2 q}{dt^2} = -\frac{1}{4} e^t \sinh q. \tag{4.1}$$

Solutions of this equation are obtained from ones appearing in Section 1.2.1 by adding $i\pi$ to $q(t)$. It is related to the radial reduction of Sine-Gordon equation considered in [49, 72, 101] by change of variable

$$q(t) = -iu\left(e^{\frac{t}{2}}\right).$$

We can describe $q(t)$ using isomonodromic deformations and Riemann-Hilbert problem in the following way. Consider the system of linear ordinary differential equations with 2 irregular singularities of Poincaré rank 1 at zero and at infinity

$$\frac{d\Phi}{dz} = A(z)\Phi(z), \quad A(z) = -\frac{ie^t\sigma_3}{16} + \frac{p\sigma_1}{2z} + \frac{1}{z^2}(i\cosh(q)\sigma_3 + \sinh(q)\sigma_2), \tag{4.2}$$

The canonical solutions at infinity are specified by the following asymptotic conditions

$$\Phi_j^{(\infty)}(z) \simeq \left(I + \sum_{m=1}^{\infty} g_m^{(\infty)} z^{-m}\right) \exp\left(-\frac{ie^t z}{16}\sigma_3\right), \qquad \text{as } z \to \infty, \quad z \in \Omega_j^{(\infty)}, \quad j = 1, \ldots, 3, \tag{4.3}$$

where the Stokes sectors are given by

$$\Omega_j^{(\infty)} = \{z : \pi(j-2) < \arg z < \pi j\}.$$

There are two Stokes matrices $S_1^{(\infty)}, S_2^{(\infty)}$ defined by the equations

$$\Phi_2^{(\infty)}(z) = \Phi_1^{(\infty)}(z)\, S_1^{(\infty)}, \quad \Phi_3^{(\infty)}(z) = \Phi_2^{(\infty)}(z)\, S_2^{(\infty)}.$$

These matrices have the triangular structure

$$S_1^{(\infty)} = \begin{pmatrix} 1 & 0 \\ s_1^{(\infty)} & 1 \end{pmatrix}, \quad S_2^{(\infty)} = \begin{pmatrix} 1 & s_2^{(\infty)} \\ 0 & 1 \end{pmatrix}.$$

The canonical solutions satisfy monodromy condition

$$\Phi_3^{(\infty)}\left(ze^{2\pi i}\right) = \Phi_1^{(\infty)}(z).$$

The canonical solutions at zero are specified by the asymptotic conditions

$$\Phi_j^{(0)}(z) \simeq G_0\Big(I + \sum_{m=1}^{\infty} g_m^{(0)} z^m\Big) \exp\left(-\frac{i}{z}\sigma_3\right), \qquad \text{as } \; z \to 0, \quad z \in \Omega_j^{(0)}, \quad j = 1, \dots, 3, \tag{4.4}$$

where

$$G_0 = i \sinh\left(\frac{q}{2}\right)\sigma_3 + i\cosh\left(\frac{q}{2}\right)\sigma_1$$

and the Stokes sectors are given by

$$\Omega_j^{(0)} = \{z : \pi(j-2) < \arg z < \pi j\}.$$

There are two Stokes matrices $S_1^{(0)}, S_2^{(0)}$ defined by the equations

$$\Phi_2^{(0)}(z) = \Phi_1^{(0)}(z)\, S_1^{(0)}, \quad \Phi_3^{(0)}(z) = \Phi_2^{(0)}(z)\, S_2^{(0)}.$$

These matrices have the triangular structure

$$S_1^{(0)} = \begin{pmatrix} 1 & s_1^{(0)} \\ 0 & 1 \end{pmatrix}, \quad S_2^{(0)} = \begin{pmatrix} 1 & 0 \\ s_2^{(0)} & 1 \end{pmatrix},$$

The canonical solutions satisfy monodromy condition

$$\Phi_3^{(0)}\left(ze^{2\pi i}\right) = \Phi_1^{(0)}(z).$$

The connection matrix C_0 relates solutions near zero and near infinity

$$\Phi_1^{(\infty)}(z) = \Phi_1^{(0)}(z)\, C_0.$$

We can notice the symmetry

$$-A(-z) = \sigma_1 A(z) \sigma_1$$

which implies

$$\Phi(-z) = \sigma_1 \Phi(z) \sigma_1 \tag{4.5}$$

We can construct from functions $\Phi_j^{(\nu)}$ the solution of the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 4.1.

- *$\Psi(z)$ is analytic for $z \in \mathbb{C} \setminus \Gamma$;*
- *$\Psi_+(z) = \Psi_-(z)J(z)$ for $z \in \Gamma$,*
- *Near the points $0, \infty$ the behavior of $\Psi(z)$ is described by (4.3), (4.4).*

Contour Γ and jump matrices $J(z)$ are given on Figure 4.1.

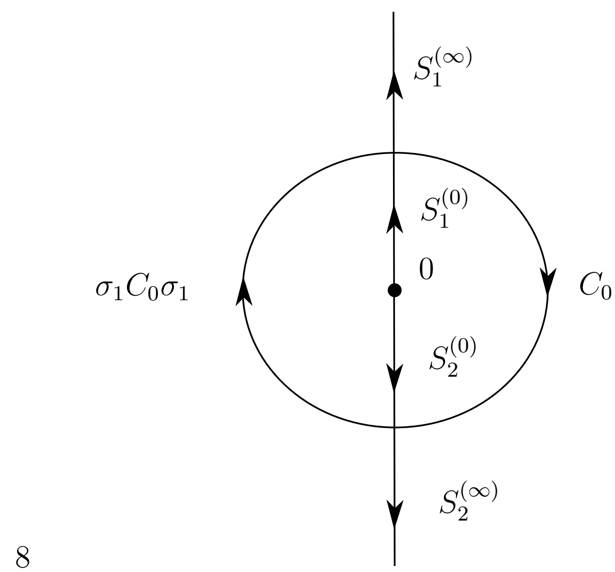

Figure 4.1. Contour Γ and jump matrices $J(z)$ for the RHP 4.1

The symmetry (4.5) now implies

$$s_1^{(0)} = s_2^{(0)}, \quad s_1^{(0)} = s_2^{(\infty)}. \tag{4.6}$$

The jump matrices also satisfy 2 cyclic relations, stating the absence of singularities at the intersections of contour Γ

$$C_0 S_1^{(\infty)} = S_1^{(0)} \sigma_1 C_0 \sigma_1, \quad \sigma_1 C_0 \sigma_1 S_2^{(\infty)} = S_2^{(0)} C_0. \tag{4.7}$$

They are related to each other through conjugation by matrix σ_1. The solution to (4.6), (4.7) can be described using complex parameters r and s and we allow r to take infinite value

$$s_1^{(0)} = s_2^{(0)} = s_1^{(0)} = s_2^{(\infty)} = s, \quad C_0 = \pm \frac{1}{\sqrt{1 + r(s-r)}} \begin{pmatrix} 1 & r \\ r-s & 1 \end{pmatrix}, \quad 1 + r(s-r) \neq 0. \tag{4.8}$$

The function $\Psi(z)$ is uniquely determined by parameter t, Stokes matrices and connection matrix. If we fix the jump matrices, then dependence of $\Psi(z)$ on t is isomonodromic. It is described by the equation

$$\frac{d\Psi}{dt} = U(z)\Psi(z), \quad U(z) = -\frac{ize^t\sigma_3}{16} + \frac{p\sigma_1}{2}. \tag{4.9}$$

Compatibility condition of (4.2) and (4.9) is given by

$$\frac{dp}{dt} = -\frac{1}{4}e^t\sinh(q), \quad \frac{dq}{dt} = p.$$

It is equation (4.1) for $q(t)$. The function $q(t)$ is given by the formula

$$q(t) \mod 2\pi i = -2\text{arcsinh}(i(G_0)_{11}).$$

The asymptotic of function $q(t)$ is presented in [71, 49, 101]. There are three types of behaviors at $-\infty$.

1. special behavior for $s = \pm 2i$

   $$q(t) \mod 2\pi i \simeq i\pi + \kappa t - 4\kappa\ln(2) + 2\kappa\ln\left(6\ln(2) - 2\gamma - \frac{2\pi}{2i + isr} - t\right), \quad t \to -\infty,$$

   where γ is the Euler's constant and $\kappa = -\text{sign}\,\text{Im}s$.

2. singular behavior for pure imaginary s with $is < 2$ or $is > 2$

   $$q(t) \bmod 2\pi i \simeq 2\kappa\ln\left(\sin\left(6\mu\ln 2 + 2\arg\Gamma(1+i\mu) - \arctan\left(\frac{\sinh(\pi\mu)}{i\cosh(\pi\mu) + \kappa r}\right) - \mu t\right)\right)$$
   $$+\kappa t - 2\kappa\ln(4\mu) + i\pi, \, t \to -\infty$$

   where

   $$\mu = \frac{1}{\pi}\text{arccosh}\left(\frac{|s|}{2}\right), \quad \mu > 0, \quad \kappa = -\text{sign}\,\text{Im}s.$$

3. generic behavior for other complex values of s

$$q(t) \mod 2\pi i = at + b + O\left(e^{t(1-|\mathrm{Re}\, a|)}\right), \quad t \to -\infty, \tag{4.10}$$

where

$$a = \frac{2}{\pi}\arcsin\left(\frac{is}{2}\right), \quad |\mathrm{Re}\, a| < 1, \quad b = \pi i - 4\pi i\eta + 6a\ln 2 - 2\ln\frac{\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)}, \tag{4.11}$$

$$\eta = \frac{1}{2\pi}\arcsin\left(\pm\sqrt{\frac{1+\frac{s^2}{4}}{1+r(s-r)}}\right), \tag{4.12}$$

$$\eta \in \left\{z : |\mathrm{Re}\, z| < \frac{1}{4}\right\} \cup \left\{z = \pm\frac{1}{4} + it : t \geq 0\right\}.$$

The choice of sign in (4.12) is the same as in (4.8). Parameters (a, b) are related to the ones in [72, 101] by

$$a = -\frac{i\alpha}{2}, \quad b = -i\beta.$$

Similarly there are three types of behaviors of function $q(t)$ at $+\infty$.

1. special behavior for $r = \infty$

$$q(t) \mod 2\pi i = i\pi - is\sqrt{\frac{2}{\pi}}\exp\left(-\frac{t}{4}\right)\exp\left(-\exp\left(\frac{t}{2}\right)\right)(1+o(1)), \quad t \to +\infty, \tag{4.13}$$

where a is given by (4.11). Solutions with such behavior are called separatrix solutions.

2. singular behavior for $1 + r(s-r) < 0$. We could not find the corresponding asymptotic formula in the literature.

3. generic behavior for $|\arg(1 + r(s-r))| < \pi$

$$\begin{aligned} q(t) \mod 2\pi i &= c^+_{0,0}\exp\left(i\exp\left(\frac{t}{2}\right) + \frac{i\nu t}{2} - \frac{t}{4}\right)\left(1 + O\left(\exp\left(-\frac{t}{2}\right)\right)\right) \\ &+ c^-_{0,0}\exp\left(-i\exp\left(\frac{t}{2}\right) - \frac{i\nu t}{2} - \frac{t}{4}\right)\left(1 + O\left(\exp\left(-\frac{t}{2}\right)\right)\right) \\ &+ O\left(\exp\left(\frac{3t}{4}(2|\mathrm{Im}\,\nu| - 1)\right)\right), \quad t \to \infty, \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} c^\pm_{0,0} &= e^{\frac{\pi\nu}{2} \mp \frac{i\pi}{4}} 2^{\pm 2i\nu}\frac{1}{\sqrt{2\pi}}\Gamma(1 \mp i\nu)(s \pm s - 2r), \\ \nu &= -\frac{1}{2\pi}\ln(1 + r(s-r)) = \frac{1}{4}c^+_{0,0}c^-_{0,0}, \quad |\mathrm{Im}\,\nu| < \frac{1}{2}. \end{aligned} \tag{4.15}$$

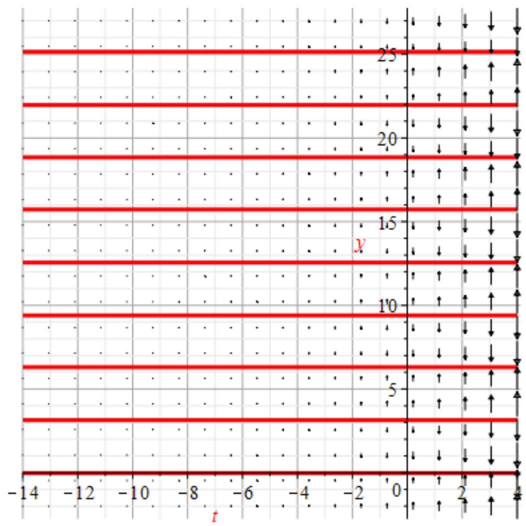

Figure 4.2. Force field for pure imaginary solutions of PIII(D8)(F) equation (4.1)

If the solution $q(t)$ is purely imaginary we have the symmetry relation

$$\overline{A(\overline{z})} = \sigma_2 A(z) \sigma_2$$

which gives

$$\overline{\Phi(\overline{z})} = \sigma_2 \Phi(z) \sigma_2, \quad \left(\overline{S_2^{(0),(\infty)}}\right)^{-1} = \sigma_2 S_1^{(0),(\infty)} \sigma_2, \quad \overline{C_0} = \sigma_2 C_0 \sigma_2, \quad s = 2\mathrm{Re}\, r.$$

Therefore purely imaginary solutions can have only generic asymptotic at $-\infty$ and generic or special asymptotic at $+\infty$. We can expect such behaviors from the analysis of force field. We denote $y(t) = iq(t)$. The force field in this case takes form $F(y,t) = -\frac{1}{4} e^t \sin y$. It creates stable trajectories $y = 2\pi k$ corresponding to generic behavior (4.14) at $+\infty$ and unstable ones $y = \pi + 2\pi k$ corresponding to the special behavior (4.13). The plot of $F(y,t)$ is depicted on Figure 4.2.

We can ask which stable trajectory solution will take if its asymptotic at $-\infty$ is given by

$$y(t) = iat + ib + O\left(e^{t(1-|\mathrm{Re}\, a|)}\right), \quad t \to -\infty.$$

To answer this question we need to determine the number of 2π in the asymptotic of $y(t)$ at $+\infty$ (see [71, 49]). First we rewrite formula (4.11) as

$$\eta = \frac{1}{2\pi} \arcsin\left(\sin\left(\frac{\pi}{2} + \frac{1}{2}\Big(ib + 6ia \ln 2\Big) + i \ln \frac{\Gamma\left(\frac{1}{2} + \frac{a}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{a}{2}\right)} \right) \right)$$

Since $q(t)$ is purely imaginary η takes real values. We can notice that values $a = b = 0$ correspond to $\eta = \frac{1}{4}$, $a = 0$, $ib = -\pi$ to $\eta = 0$, and $a = 0$, $ib = -2\pi$ to $\eta = -\frac{1}{4}$. Since the asymptotic of $y(t)$ depends continuously on (a, b), the number of 2π in the asymptotic of $q(t)$ at $+\infty$ can be described using parameter ζ

$$\zeta = \frac{1}{4} + \frac{1}{4\pi}\Big(ib + 6ia\ln 2\Big) + \frac{i}{2\pi}\ln\frac{\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)}$$

If $\frac{k}{2} < \zeta < \frac{k+1}{2}$, asymptotic of $y(t)$ at $+\infty$ is generic and solution lies in the interval $((2k-1)\pi, (2k+1)\pi)$ for large t. If $\zeta = \frac{k}{2}$, asymptotic of $y(t)$ at $+\infty$ is special and solution approaches $(2k-1)\pi$ for large t. As the result the formulae (4.14), (4.13) are modified as

$$y(t) = (2k-1)\pi - 2i\sin\left(\frac{\pi a}{2}\right)\sqrt{\frac{2}{\pi}}\exp\left(-\frac{t}{4}\right)\exp\left(-\exp\left(\frac{t}{2}\right)\right)(1+o(1)), \quad t \to +\infty,$$

$$y(t) = 2\pi k + 4\sqrt{-\nu}\ \exp\left(-\frac{t}{4}\right)\cos\left(\exp\left(\frac{t}{2}\right) + \frac{\nu t}{2} + \phi\right)\left(1 + O\left(\exp\left(-\frac{t}{2}\right)\right)\right)$$
$$+O\left(\exp\left(\frac{3t}{4}(2|\mathrm{Im}\,\nu| - 1)\right)\right), \quad t \to \infty,$$

where

$$\phi = 2\nu\ln 2 + \frac{3\pi}{4} - \arg(\Gamma(i\nu)) - \arg(r), \quad k = \lfloor 2\zeta \rfloor,$$
$$\sigma = \frac{1}{4} - \frac{a}{4}, \quad \nu = \frac{1}{2\pi}\ln\frac{\sin^2 2\pi\eta}{\sin^2 2\pi\sigma}, \quad r = -i\frac{\sin 2\pi(\sigma+\eta)}{\sin 2\pi\eta}. \tag{4.16}$$

These connection formulae are illustrated on Figures 4.3 and 4.4.

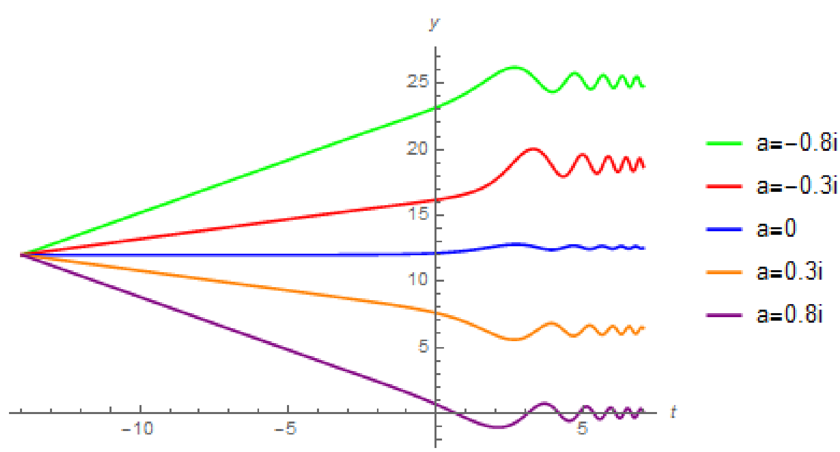

Figure 4.3. Pure imaginary solutions of PIII(D8)(F) equation (4.1) attaining stable trajectories

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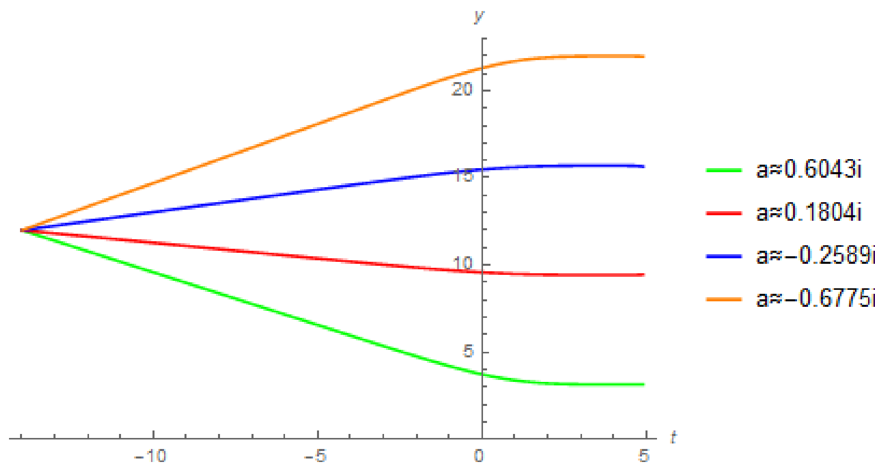

Figure 4.4. Pure imaginary solutions of PIII(D8)(F) equation (4.1) attaining unstable trajectories

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If the solution $q(t)$ is real we have the symmetry relation

$$\overline{A(\overline{z})} = \sigma_1 A(z)\sigma_1 \tag{4.17}$$

which gives

$$\overline{\Phi(\overline{z})} = \sigma_1\Phi(z)\sigma_1, \quad \left(\overline{S_2^{(0),(\infty)}}\right)^{-1} = \sigma_1 S_1^{(0),(\infty)}\sigma_1, \quad \overline{C_0} = \sigma_1 C_0\sigma_1, \quad s = 2i\,\mathrm{Im}\,r. \tag{4.18}$$

For solutions taking values in $\mathbb{R} + i\pi k$ we also have relations (4.17), (4.18).

For real solutions of (4.1) there is only one stable trajectory $q = 0$. The corresponding force field is given on Figure 4.5. Such solutions can have only generic behavior at $-\infty$ and $+\infty$. The asymptotic (4.14) can be written as

$$y(t) = 4\sqrt{\nu}\,\exp\left(-\frac{t}{4}\right)\sin\left(\exp\left(\frac{t}{2}\right) + \frac{\nu t}{2} + \phi\right)\left(1 + O\left(\exp\left(-\frac{t}{2}\right)\right)\right)$$
$$+O\left(\exp\left(\frac{3t}{4}(2|\mathrm{Im}\,\nu| - 1)\right)\right), \quad t \to +\infty,$$

where ϕ, ν and r are given by (4.16). The typical solution is shown on Figure 4.6.

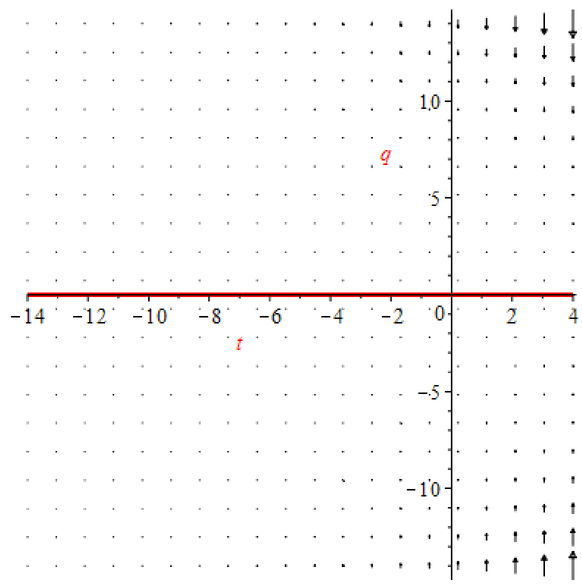

Figure 4.5. Force field for real solutions of PIII(D8)(F) equation (4.1).

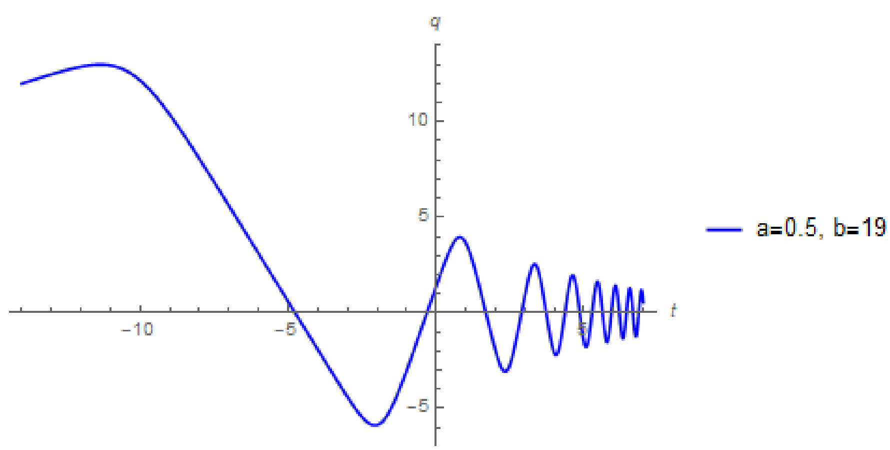

Figure 4.6. Real solution of PIII(D8)(F) equation (4.1)

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For solutions taking values in $\mathbb{R} - i\pi - 2\pi ik$ there is one nonstable trajectory $q = -i\pi - 2\pi ik$. The force field for $y = q + i\pi + 2\pi ik$ is given on Figure 1.6. Such solutions can have all three types of behaviors at $-\infty$ and special or singular behavior at $+\infty$. Solutions with special behavior at $+\infty$ and generic or special behavior at $-\infty$ appeared in the application to the Ising model as (1.26), (1.29).

Let us turn our attention now to the tau function. It is given by the formula

$$\tau(t_1, t_2, \sigma, \eta) = \exp\left(\int_{t_1}^{t_2} H dt\right). \tag{4.19}$$

where the Hamiltonian H is given by

$$H(p, q, t) = \frac{p^2}{2} + \frac{e^t \cosh(q)}{4}, \quad p = \frac{dq}{dt}. \tag{4.20}$$

It is the square of tau function considered in [72].

The connection constant for tau function corresponding to solutions with special behavior at $+\infty$ and generic or special behavior at $-\infty$ was found in [110], as we mentioned in Section 1.2.1.

The connection constant for tau function corresponding to solutions with generic behaviors at $-\infty$ and $+\infty$ was conjectured in [70] and proven in [72]. We will obtain this formula below.

It is convenient to take σ and η as the independent parameters. Numbers a and b are then given by

$$a = 1 - 4\sigma, \quad b = \pi i - 4\pi i\eta - (2 - 8\sigma)\ln 8 - 2\ln\frac{\Gamma(1-2\sigma)}{\Gamma(2\sigma)}, \tag{4.21}$$

where σ, η are the complex numbers satisfying

$$\begin{gathered} \eta \in \left\{z : |\mathrm{Re}\, z| < \frac{1}{4}\right\} \cup \left\{z = \pm\frac{1}{4} + it : \, t \geq 0\right\} \\ \eta \neq 0, \quad \left|\arg\frac{\sin^2 2\pi\eta}{\sin^2 2\pi\sigma}\right| < \pi, \quad 0 < \mathrm{Re}\,\sigma < \frac{1}{2} \end{gathered} \tag{4.22}$$

We rewrite expressions of the asymptotic parameters at $t = +\infty$ (4.15) in terms of σ and η

$$c^{\pm}_{0,0} = ie^{\frac{\pi\nu}{2} \mp \frac{i\pi}{4}} 2^{1\pm 2i\nu} \frac{1}{\sqrt{2\pi}}\Gamma(1 \mp i\nu)\frac{\sin 2\pi(\sigma \mp \eta)}{\sin 2\pi\eta}, \quad \nu = \frac{1}{2\pi}\ln\frac{\sin^2 2\pi\eta}{\sin^2 2\pi\sigma}. \tag{4.23}$$

If we look in more details on asymptotic (4.14) we can notice the following structure (see [70]),

$$q\,(t) \mod 2\pi i \simeq \sum_{l,k\geq 0,\, \epsilon=\pm} c^{\epsilon}_{k,l} r^{\epsilon(2k+1)} \zeta^{2l+2k+1}. \tag{4.24}$$

$$r = \exp\left(i\exp\left(\frac{t}{2}\right) + \frac{i\nu t}{2}\right), \; \zeta = \exp\left(-\frac{t}{4}\right); \; c^{\pm}_{0,1} = \pm\frac{ic^{\pm}_{0,0}}{8}(6\nu^2 \pm 4i\nu - 1), \; c^{\pm}_{1,0} = \frac{1}{48}(c^{\pm}_{0,0})^3$$

Justification of asymptotic (4.24), can be done using the Riemann-Hilbert approach, cf [38]. Using the fact that $|\mathrm{Im}\,\nu| < \frac{1}{2}$ we can notice that formula (4.24) is actually asymptotic expansion. Substituting it and formula (4.10) at the right hand side of equation (4.19), we arrive at the following asymptotic representation of the tau function (4.19) as $t_1 \to -\infty$, $t_2 \to +\infty$

$$\ln\tau(t_1,t_2,\sigma,\eta) \simeq \frac{e^{t_2}}{4} + 4\nu e^{\frac{t_2}{2}} + \frac{\nu^2 t_2}{2} - \frac{a^2 t_1}{2} + \ln\Upsilon \tag{4.25}$$

Our goal is to evaluate constant term in the asymptotics, which we called Υ.

In Theorem 3.1 we showed the following identity

$$\ln\tau(t_1,t_2,\sigma,\eta) = 2H|_{t_1}^{t_2} + S\left(t_1,t_2,\frac{1}{4},\frac{1}{4}\right) + \int\limits_{\left(\frac{1}{4},\frac{1}{4}\right)}^{(\sigma,\eta)} \left(p\frac{\partial q}{\partial\sigma}d\sigma + p\frac{\partial q}{\partial\eta}d\eta\right)\Bigg|_{t_1}^{t_2}, \tag{4.26}$$

We picked the reference point $\sigma = \eta = \frac{1}{4}$, which corresponds to solution $q(t) \equiv 0$. Therefore

$$S\left(t_1,t_2,\frac{1}{4},\frac{1}{4}\right) = \frac{1}{4}(e^{t_1} - e^{t_2}).$$

Substituting (4.24) and (4.10) at the right hand side of equation (4.26) and comparing the result with (4.25), we shall arrive, after rather tedious though straightforward calculations, at the following formula for Υ

$$\ln\Upsilon = 2\nu^2 - a^2 - 2i\nu + \int\limits_{\left(\frac{1}{4},\frac{1}{4}\right)}^{(\sigma,\eta)} (ic^+_{0,0}dc^-_{0,0} - adb). \tag{4.27}$$

Following [70], we introduce the parameter ρ

$$e^{-4\pi i\rho} = \frac{\sin 2\pi(\sigma+\eta)}{\sin 2\pi\eta}. \tag{4.28}$$

It is determined up to a half-integer, which will not affect our calculations. Using connection formulae (4.23), (4.21), we can re-write the differential form $(c^+_{0,0}dc^-_{0,0} - adb)$ as the differential form in variables η, ρ, σ and ν,

$$c^+_{0,0}dc^-_{0,0} - adb = 4\nu d(\ln c^-_{0,0}) - adb = -16\pi i(\sigma d\eta + i\nu d\rho) + 4\pi i d\eta - (24 - 96\sigma)(\ln 2)d\sigma$$

$$+(2\pi i + 8\ln 2)\nu d\nu + (2 - 8\sigma)d\ln\frac{\Gamma(1-2\sigma)}{\Gamma(2\sigma)} + 4i\nu d\ln\Gamma(1+i\nu).$$

After that we re-write (4.27) as

$$\ln\Upsilon = 2\nu^2 - 2i\nu + i\pi\nu^2 + 4\nu^2\ln 2 - 16\sigma^2 + 8\sigma - 1 - 24\sigma\ln 2 + 48\sigma^2\ln 2 + 3\ln 2 + 4\pi i\eta$$
$$-\pi i - 16\pi i\int\limits_{\left(\frac{1}{4},\frac{1}{4}\right)}^{(\sigma,\eta)}(\sigma d\eta + i\nu d\rho) + \int\limits_{\frac{1}{4}}^{\sigma}(2-8\sigma)d\ln\frac{\Gamma(1-2\sigma)}{\Gamma(2\sigma)} + \int\limits_{0}^{\nu}4i\nu d\ln\Gamma(1+i\nu). \tag{4.29}$$

It remains to evaluate the integrals in (4.29). For the integrals involving the Γ-functions one gets,

$$\int\limits_{\frac{1}{4}}^{\sigma}(2-8\sigma)d\ln\frac{\Gamma(1-2\sigma)}{\Gamma(2\sigma)} = 2\ln\frac{\Gamma(1-2\sigma)}{\Gamma(2\sigma)} - 8\sigma + 16\sigma^2 + 1$$
$$+4\ln\Big(G(1-2\sigma)G(1+2\sigma)\Big) - 8\ln G\left(\frac{1}{2}\right) - 4\ln\Gamma\left(\frac{1}{2}\right), \tag{4.30}$$

$$\int\limits_{0}^{\nu}4i\nu d\ln\Gamma(1+i\nu) = 2\nu i - 2\nu^2 - 2i\nu\ln(2\pi) + 4\ln G(1+i\nu), \tag{4.31}$$

where $G(z)$ is the Barnes G-function and we have used the classical formula,

$$\int\limits_{0}^{z}\ln\Gamma(x)dx = \frac{z(1-z)}{2} + \frac{z}{2}\ln(2\pi) + z\ln\Gamma(z) - \ln G(1+z). \tag{4.32}$$

The branch of the logarithms is fixed by the requirement of the expression to be real for positive z.

Evaluation of the first integral in (4.29) is more challenging. It was done in [70]. First of all we integrate by parts

$$\int\limits_{\left(\frac{1}{4},\frac{1}{4}\right)}^{(\sigma,\eta)}\sigma d\eta + i\nu d\rho = \sigma\eta + i\nu\rho - \int\limits_{\left(\frac{1}{4},\frac{1}{4}\right)}^{(\sigma,\eta)}\eta d\sigma + i\rho d\nu - \frac{1}{16}.$$

To fix the constant we noticed that when $\sigma\to\frac{1}{4},\quad \eta\to\frac{1}{4}$, we have $\rho\to -i\infty,\quad \nu\to 0,\quad \nu\rho\to 0$.

We can notice that the Hamiltonian (4.20) is invariant under transformation $q\to -q$. It corresponds to transformations $\eta\to\frac{1}{2}-\eta$ and $\sigma\to\frac{1}{2}-\sigma$. Using that we can assume without loss of generality that

$$0 < \operatorname{Re}\eta < \frac{1}{4}$$

Now parameters ρ and η can be considered as functions of (σ,ν):

$$\eta = \frac{1}{2\pi}\arcsin\left(\pm\sqrt{e^{2\pi\nu}\sin^2 2\pi\sigma}\right),$$

and ρ is described by (4.28). We have

$$\int_{(\frac{1}{4},\frac{1}{4})}^{(\sigma,\eta)} \eta d\sigma + i\rho d\nu = -\frac{\eta^2}{2}+\frac{\nu^2}{8}+\frac{1}{8\pi^2}\mathrm{Li}_2(-e^{2\pi i(\sigma+\eta-i\frac{\nu}{2})})+\frac{1}{8\pi^2}\mathrm{Li}_2(-e^{-2\pi i(\sigma+\eta+i\frac{\nu}{2})})-\frac{1}{96}, \quad (4.33)$$

where $\mathrm{Li}_2\,(z)$ denotes the dilogarithm function and we remind the integral formula for it

$$\mathrm{Li}_2\,(z) = -\int_0^z \frac{\ln(1-x)}{x}dx. \quad (4.34)$$

To check (4.33) we notice the formula

$$2\cos\pi(\sigma+\eta\pm\frac{i\nu}{2}) = e^{i\pi(\pm\sigma\mp\eta-\frac{i\nu}{2}-4\rho)}, \quad (4.35)$$

which implies

$$\ln\left(1+e^{2\pi i\left(\sigma+\eta-\frac{i\nu}{2}\right)}\right)-\ln\left(1+e^{-2\pi i\left(\sigma+\eta+\frac{i\nu}{2}\right)}\right) = 4\pi i\eta \quad (4.36)$$

$$\ln\left(1+e^{2\pi i\left(\sigma+\eta-\frac{i\nu}{2}\right)}\right)+\ln\left(1+e^{-2\pi i\left(\sigma+\eta+\frac{i\nu}{2}\right)}\right) = 2\pi\nu-8\pi i\rho \quad (4.37)$$

We see using (4.36), (4.37) that derivatives of left and right hand side in (4.33) coincide. We also remind that

$$\mathrm{Li}_2(1) = \frac{\pi^2}{6}.$$

We remind yet another classical formula,

$$\mathrm{Li}_2(e^{2\pi iz}) = -2\pi i\ln\hat{G}(z) - 2\pi iz\ln\frac{\sin(\pi z)}{\pi} - \pi^2 z(1-z) + \frac{\pi^2}{6}, \quad (4.38)$$

$$\mathrm{Li}_2(e^{-2\pi iz}) = 2\pi i\ln\hat{G}(z) + 2\pi iz\ln\frac{\sin(\pi z)}{\pi} - \pi^2 z(1-z) + \frac{\pi^2}{6}, \quad (4.39)$$

where

$$\hat{G}(z) = \frac{G(1+z)}{G(1-z)}. \quad (4.40)$$

The branches of logarithms are fixed by the requirement that $\ln\hat{G}(z)$ and $\ln\sin(\pi z)$ are real for $0 < z < 1$.

Taking into account the relations (4.33), (4.35), (4.38), (4.39) we arrive at the following final expression for the first integral in (4.29)

$$-16\pi i \int\limits_{(\frac{1}{4},\frac{1}{4})}^{(\sigma,\eta)} \sigma d\eta + i\nu d\rho = 4\ln\frac{\hat{G}(\sigma+\eta+\frac{1-i\nu}{2})}{\hat{G}(\sigma+\eta+\frac{1+i\nu}{2})} - 8\pi i\sigma\eta + 4\pi i\eta^2 - i\pi\nu^2$$
$$+4i\nu\ln(2\pi) - 4\pi i\sigma^2 - 4\pi i\sigma + 4\pi i\eta + \frac{\pi i}{2}. \tag{4.41}$$

Substituting formulae (4.30), (4.31), and (4.41) in (4.29) we arrive at the main result of this section

Theorem 4.1 ([72])**.** *Let σ and η be the "monodromy" parameters of the solution $q(t)$ of Painlevé III(D8)(F) equation* (4.1) *satisfying the inequalities (4.22). Then the tau function* (4.19) *has the behavior* (4.25) *as $t_1 \to -\infty$, $t_2 \to +\infty$ with*

$$\Upsilon = (2\pi)^{2i\nu} 2^{4\nu^2+48\sigma^2-24\sigma} e^{4\pi i(\eta^2-2\sigma\eta-\sigma^2+2\eta-\sigma)} \left(\frac{\Gamma(1-2\sigma)}{\Gamma(2\sigma)}\right)^2$$
$$\left(\frac{G(1+i\nu)G(1+2\sigma)G(1-2\sigma)\hat{G}(\sigma+\eta+\frac{1-i\nu}{2})}{\hat{G}(\sigma+\eta+\frac{1+i\nu}{2})}\right)^4 \frac{(-8i)}{\pi^2(G(\frac{1}{2}))^8}, \tag{4.42}$$

where ν is defined in (4.23), $G(z)$ is the Barnes G – function, and $\hat{G}(z)$ is given by (4.40)*.*

Remark 4.1. *If* $\sin 2\pi(\sigma+\eta) = 0$ *then parameter ρ is undefined. But we can notice that the answer* (4.42) *does not depend on ρ and is holomorphic in σ and ν, therefore it stays the same if* $\sin 2\pi(\sigma+\eta) = 0$.

Remark 4.2. *It was shown in [70, 72] that the variables* $(8\pi i\sigma, \eta)$ *and* $(8\pi\nu, \rho)$ *are Darboux coordinates for the symplectic form given by differential $d\omega$ of form* (2.16)*. In fact, one has that*

$$d\omega = 8\pi i d\sigma \wedge d\eta = 8\pi d\nu \wedge d\rho.$$

That shows Conjecture 2.1 in the setting of system (4.2)*.*

4.2 Homogeneous PII equation

We are concerned with the solutions of homogeneous second Painlevé equation

$$\frac{d^2q}{dt^2} = 2q^3 + tq \tag{4.43}$$

Its interpretation as equation of isomonodromic deformations was given in Section 1.1.1. The solutions can be parametrized by the set

$$\{(s_1, s_2, s_3) : (1.9)\,\text{holds}\}.$$

The respective asymptotics and their explicit monodromy parametrization are presented in [79], [71], [38], and [49]. There are three types of behaviors at $-\infty$.

1. special behavior for $1 - s_1 s_3 = 0$

$$q(t) \simeq \sigma\sqrt{\frac{-t}{2}} \sum_{n=0\infty} b_n(-t)^{-\frac{3n}{2}} - \frac{s_1 + s_2}{\sqrt{\pi}2^{\frac{7}{4}}(-t)^{\frac{1}{4}}} \exp\left(-\frac{2\sqrt{2}}{3}(-t)^{\frac{3}{2}}\right)(1+O((-t)^{-\frac{1}{4}})),\; t \to -\infty$$

   where $s_1 = -i\sigma, \quad \sigma = \pm 1$. Solutions with such behavior are called increasing tronqué solutions or separatrix solutions.

2. singular behavior for $1 - s_1 s_3 < 0$

$$q(t) = \frac{2\sqrt{-t}}{ae^{ig} + be^{-ig} + O((-t)^{-\frac{3}{10}})}$$

   where

$$a = \frac{\sqrt{2\pi}e^{\frac{\pi\beta}{2}}}{s_1\Gamma\left(\frac{1}{2} + i\beta\right)}, \quad b = \frac{\sqrt{2\pi}e^{\frac{\pi\beta}{2}}}{s_3\Gamma\left(\frac{1}{2} - i\beta\right)}, \quad ab = 1,$$

$$g = \frac{2}{3}(-t)^{\frac{3}{2}} + \frac{3\beta}{2}\ln(-t) + 3\beta\ln 2 - \frac{\pi}{2}, \quad \beta = \frac{1}{2\pi}\ln(s_1 s_3 - 1).$$

3. generic behavior for $|\arg(1 - s_1 s_3)| < \pi$

$$q(t) = a^+_{0,0}e^{\frac{2i}{3}(-t)^{\frac{3}{2}}}(-t)^{\frac{3\mu}{2}-\frac{1}{4}} + a^-_{0,0}e^{-\frac{2i}{3}(-t)^{\frac{3}{2}}}(-t)^{-\frac{3\mu}{2}-\frac{1}{4}} + O\left(|t|^{\frac{9|\mathrm{Re}\mu|}{2}-\frac{7}{4}}\right), \quad t \to -\infty, \tag{4.44}$$

$$\mu = -\frac{\ln(1 - s_1 s_3)}{2\pi i}, \qquad a^+_{0,0}a^-_{0,0} = \frac{i\mu}{2},$$
$$a^+_{0,0} = \frac{\sqrt{\pi}\,2^{3\mu}e^{-\frac{i\pi\mu}{2}-\frac{i\pi}{4}}}{s_1\Gamma(\mu)}, \qquad a^-_{0,0} = \frac{\sqrt{\pi}\,2^{-3\mu}e^{-\frac{i\pi\mu}{2}+\frac{i\pi}{4}}}{s_3\Gamma(-\mu)}, \tag{4.45}$$

There are also three types of behaviors at $+\infty$.

1. special behavior for $s_2 = 0$

$$q(t) \simeq \frac{is_1}{2\sqrt{\pi}t^{\frac{1}{4}}} \exp\left(-\frac{2}{3}t^{\frac{3}{2}}\right)(1 + O(t^{-\frac{3}{4}})),\ t \to -\infty.$$

Solutions with such behavior are called decreasing tronqué solutions or separatrix solutions.

2. singular behavior for $s_2 \in \mathbb{R}, \quad s_2 \neq 0$

$$q(t) = \frac{i\varepsilon}{\sqrt{2}}\left(\frac{ce^{ih} - 1 + O(t^{-\frac{3}{4}})}{ce^{ih} + 1 + O(t^{-\frac{3}{4}})}\right) + O(t^{-\frac{3}{2}})$$

where

$$c = \frac{\sqrt{2\pi}e^{\frac{\pi\gamma}{2}}}{(1 + s_2 s_3)\Gamma\left(\frac{1}{2} + i\gamma\right)}, \quad h = \frac{2\sqrt{2}}{3}t^{\frac{3}{2}} + \frac{3\gamma}{2}\ln t + \frac{7\gamma}{2}\ln 2, \quad \gamma = \frac{1}{\pi}\ln(\varepsilon s_2), \quad \varepsilon = \mathrm{sign} s_2.$$

3. generic behavior for $|\arg(i\sigma s_2)| < \frac{\pi}{2}, \quad \sigma = \mathrm{sign}\,\mathrm{Re}\,(is_2) = \pm 1$

$$\sigma q\,(t) = i\sqrt{\frac{t}{2}} + b^{+}_{1,1}e^{\frac{2i\sqrt{2}}{3}t^{\frac{3}{2}}}t^{-\frac{3\nu}{2}-\frac{1}{4}} + b^{-}_{1,1}e^{-\frac{2i\sqrt{2}}{3}t^{\frac{3}{2}}}t^{\frac{3\nu}{2}-\frac{1}{4}} + O\left(t^{3|\mathrm{Re}\nu|-1}\right), \qquad t \to +\infty, \tag{4.46}$$

$$\nu = \frac{\ln(i\sigma s_2)}{\pi i}, \qquad b^{+}_{1,1}b^{-}_{1,1} = \frac{i\nu}{4\sqrt{2}},$$
$$b^{+}_{1,1} = \frac{\sqrt{\pi}\,2^{-\frac{7\nu}{2}-\frac{3}{4}}e^{\frac{i\pi\nu}{2}-\frac{i\pi}{4}}}{(1 + s_2 s_3)\Gamma(-\nu)}, \qquad b^{-}_{1,1} = -\frac{\sqrt{\pi}\,2^{\frac{7\nu}{2}-\frac{3}{4}}e^{\frac{i\pi\nu}{2}+\frac{i\pi}{4}}}{(1 + s_1 s_2)\Gamma(\nu)}. \tag{4.47}$$

Pure imaginary solutions of (4.43) were considered in Section 1.1.1. They have generic behavior at $-\infty$ and special or generic behavior at $+\infty$.

For the real solutions the symmetry conditions (1.6), (1.7) are replaced by

$$\overline{A(\overline{z})} = \sigma_1 A(z)\sigma_1.$$

and

$$\overline{\Psi(\overline{z})} = \sigma_1\Phi(z)\sigma_1, \quad \left(\overline{S_n}\right)^{-1} = \sigma_1 S_{7-n}\sigma_1, \quad s_3 = \overline{s_1}.$$

The force field for real solutions of (4.43) is given on Figure 4.7. It provides unstable trajectory $q = 0$ for positive time corresponding to special behavior and three trajectories

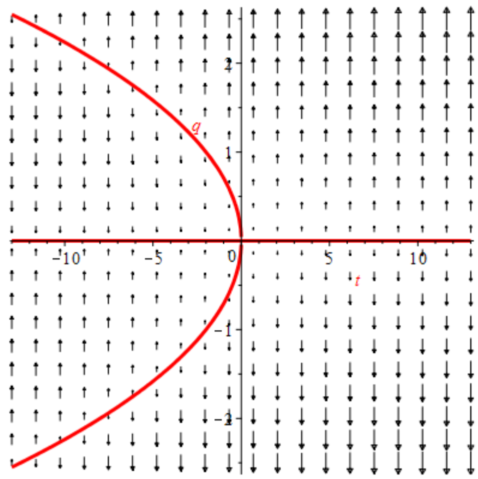

Figure 4.7. Force field for real solutions of homogeneous PII equation

for negative time: stable $q = 0$ corresponding to generic behavior and unstable $q = \pm\sqrt{\frac{-t}{2}}$ corresponding to special behavior. Real solutions can have all three types of behavior at $-\infty$ and special or singular behavior at $+\infty$.

If $-1 < is_1 < 1$, then the real solution has generic behavior at $-\infty$ and special behavior at $+\infty$. Such solutions are called Ablowitz-Segur solutions. The formula (4.44) can be rewritten in this case as

$$q(t) = \frac{d}{(-t)^{\frac{1}{4}}} \cos\left(\frac{2}{3}(-t)^{\frac{3}{2}} - \frac{3}{4}d^2 \ln(-t) + \phi\right) + O\left(\frac{1}{|t|}\right), \quad t \to -\infty,$$

where

$$d = \sqrt{-\frac{1}{\pi}\ln\left(1 - |s_1|^2\right)}, \quad \phi = -\frac{\pi}{4} - \frac{3}{2}d^2 \ln 2 + \arg\left(\Gamma\left(i\frac{d^2}{2}\right)\right) - \arg(s_1).$$

If $s_1 = \pm i$ then the real solution has special behavior at $-\infty$ and special behavior at $+\infty$. Such solution is called Hastings-Mcleod solution.

The behavior of Ablowitz-Segur and Hastings-Mcleod solutions is illustrated on Figure 4.8.

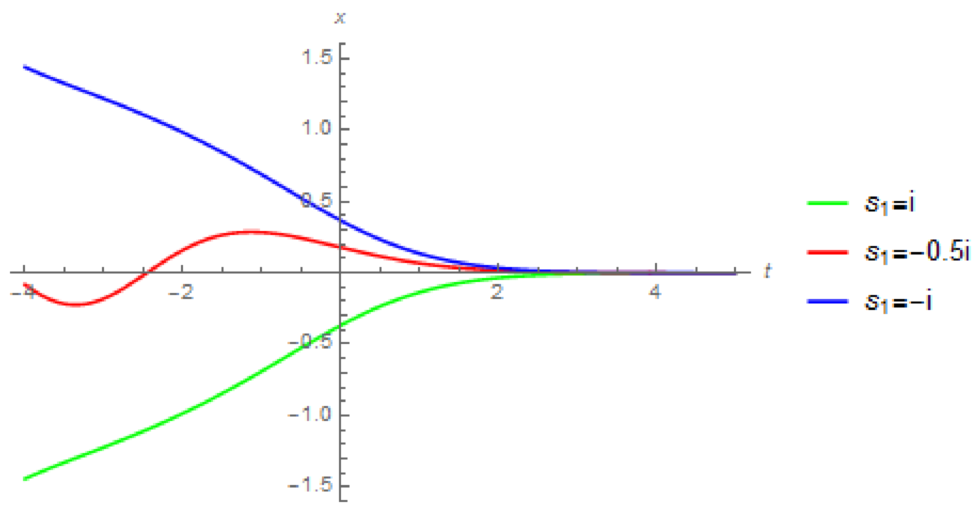

Figure 4.8. Real solutions of homogeneous PII equation

Let us return now to the tau function. It is given by the formula

$$\tau(t_1, t_2, s_1, s_2) = \exp\left(\int_{t_1}^{t_2} H dt\right). \tag{4.48}$$

where the Hamiltonian H is given by

$$H(p, q, t) = \frac{p^2}{2} - \frac{q^4}{2} - \frac{tq^2}{2}, \quad p = \frac{dq}{dt}. \tag{4.49}$$

It is the square root of tau function considered in [69].

The tau function connection constant for Ablowitz-Segur solutions was conjectured in [17] and computed in [22, 21]. The tau function connection constant for Hastings-Mcleod solution was found in [33, 4].

The tau function connection constant for solutions with generic asymptotics (4.44) at $-\infty$ and (4.46) at $+\infty$ was computed in [69]. We provide the derivation of this formula below.

Generic asymptotics at $-\infty$ and $+\infty$ happens under conditions

$$|\arg(1 - s_1 s_3)| < \pi, \quad |\arg(i\sigma s_2)| < \frac{\pi}{2}, \quad \sigma = \operatorname{sign} \operatorname{Re}(is_2) = \pm 1, \tag{4.50}$$

which imply using (4.45) and (4.47)

$$|\text{Re}\mu| < \frac{1}{2}, \quad |\text{Re}\nu| < \frac{1}{2}. \tag{4.51}$$

We will need more terms in the asymptotics of $q(t)$ for our calculations. Denote

$$r = e^{\frac{2i}{3}(-t)^{\frac{3}{2}}}(-t)^{\frac{3\mu}{2}}, \qquad \zeta = (-t)^{-\frac{1}{4}}.$$

We have the following formal asymptotic expansion at $t = -\infty$:

$$q(t) \simeq \sum_{l \geq k \geq 0,\, \epsilon = \pm} a_{k,l}^{\epsilon} r^{\epsilon(2k+1)} \zeta^{6l+1}. \tag{4.52}$$

A few first terms are

$$q(t) \simeq \left(a_{0,0}^{+} r + a_{0,0}^{-} r^{-1}\right)\zeta + \left(a_{0,1}^{+} r + a_{0,1}^{-} r^{-1} + a_{1,1}^{+} r^{3} + a_{1,1}^{-} r^{-3}\right)\zeta^{7} + \ldots,$$

where

$$a_{0,1}^{\pm} = \frac{i a_{0,0}^{\pm}\left(\mp 102\mu^2 + 36\mu \mp 5\right)}{48}, \qquad a_{1,1}^{\pm} = -\frac{\left(a_{0,0}^{\pm}\right)^3}{4}.$$

Similarly, denoting

$$y = e^{\frac{2i\sqrt{2}}{3} t^{\frac{3}{2}}} t^{-\frac{3\nu}{2}}, \qquad \xi = t^{-\frac{1}{4}},$$

we have a formal asymptotic expansion at $t = +\infty$:

$$q(t) \simeq \sum_{l \geq k \geq 0,\, \epsilon = \pm} b_{2k+1,6l+1}^{\epsilon} y^{\epsilon(2k+1)} \xi^{6l+1} + \sum_{l \geq k \geq 0,\, \epsilon = \pm} b_{2k,6l-2}^{\epsilon} y^{2\epsilon k} \xi^{6l-2}. \tag{4.53}$$

Let us record its several first terms:

$$\begin{aligned} \sigma q(t) \simeq{}& \frac{i\xi^{-2}}{\sqrt{2}} + \left(b_{1,1}^{+} y + b_{1,1}^{-} y^{-1}\right)\xi + \left(b_{0,4} + b_{2,4}^{+} y^{2} + b_{2,4}^{-} y^{-2}\right)\xi^{4} \\ &+ \left(b_{1,7}^{+} y + b_{1,7}^{-} y^{-1} + b_{3,7}^{+} y^{3} + b_{3,7}^{-} y^{-3}\right)\xi^{7} + \\ &+ \left(b_{0,10} + b_{2,10}^{+} y^{2} + b_{2,10}^{-} y^{-2} + b_{4,10}^{+} y^{4} + b_{4,10}^{-} y^{-4}\right)\xi^{10} + \ldots, \end{aligned}$$

where

$$b_{0,4} = -\frac{3\nu}{4}, \qquad b_{2,4}^{\pm} = -\frac{i\sqrt{2}}{2}\left(b_{1,1}^{\pm}\right)^2,$$

$$b_{1,7}^{\pm} = b_{1,1}^{\pm}\frac{i\sqrt{2}}{6}\left(\mp\frac{51}{8}\nu^2 - \frac{3}{2}\nu \mp \frac{17}{16}\right), \qquad b_{3,7}^{\pm} = -\frac{\left(b_{1,1}^{\pm}\right)^3}{2},$$

$$b_{0,10} = \frac{i\sqrt{2}}{2}\left(\frac{51}{32}\nu^2 + \frac{1}{8}\right), \quad b_{2,10}^{\pm} = \left(\mp\frac{17}{8}\nu^2 - \frac{11}{8}\nu \mp \frac{41}{48}\right)\left(b_{1,1}^{\pm}\right)^2, \quad b_{4,10}^{\pm} = \frac{i\sqrt{2}}{4}\left(b_{1,1}^{\pm}\right)^4.$$

Justification of asymptotics (4.52), (4.53) can be done using the Riemann-Hilbert approach, cf [38]. Conditions (4.51) imply that these expressions are actually asymptotic expansions. Plugging them into (4.48), we get the following behavior as $t_1 \to -\infty$, $t_2 \to +\infty$

$$\ln \tau(t_1, t_2, s_1, s_2) \simeq \frac{t_2^3}{24} + \frac{i\sqrt{2}}{3}\nu t_2^{\frac{3}{2}} - \frac{(6\nu^2+1)}{16}\ln t_2 + \frac{2i\mu}{3}(-t_1)^{\frac{3}{2}} + \frac{3\mu^2}{4}\ln(-t_1) + \ln \Upsilon \tag{4.54}$$

Our goal is to evaluate Υ.

In Theorem 3.1 we showed the following identity

$$\ln \tau(t_1, t_2, s_1, s_2) = \left(\frac{2tH}{3} - \frac{pq}{3}\right)\Bigg|_{t_1}^{t_2} + S(t_1, t_2, s_1, s_2).$$

Taking derivative with respect to monodromy data s_1, s_2 we have

$$\frac{\partial}{\partial s_j}\ln \tau(t_1, t_2, s_1, s_2) = \frac{\partial}{\partial s_j}\left(\frac{2tH}{3} - \frac{pq}{3}\right)\Bigg|_{t_1}^{t_2} + p\frac{\partial q}{\partial s_j}\Bigg|_{t_1}^{t_2}.$$

Therefore we can write the following representation for tau function

$$\begin{aligned}\ln \tau(t_1, t_2, s_1, s_2) = \ln \tau(t_1, t_2, 0, -i) + \left(\left(\frac{2tH}{3} - \frac{pq}{3}\right)\Bigg|_{t_1}^{t_2}\right)\Bigg|_{(0,-i)}^{(s_1,s_2)} \\ + \int\limits_{(0,-i)}^{(s_1,s_2)} p\frac{\partial q}{\partial s_1}\Bigg|_{t_1}^{t_2} ds_1 + p\frac{\partial q}{\partial s_2}\Bigg|_{t_1}^{t_2} ds_2\end{aligned} \tag{4.55}$$

We picked the reference point $s_1 = 0$, $s_2 = -i$ since it can be related to Hastings-Mcleod solution of (4.43) (see Section 4.2.1). It corresponds to $\nu = \mu = 0$. As a particular case of (4.54) we have

$$\ln \tau(t_1, t_2, 0, -i) \sim \frac{t_2^3}{24} - \frac{1}{16}\ln t_2 + \ln \Upsilon_0. \tag{4.56}$$

We will compute Υ_0 in Section 4.2.1.

Now plugging (4.52), (4.53), and (4.56) in (4.55) and comparing the result with (4.54) we get

$$\ln \Upsilon = \ln \Upsilon_0 - \frac{\nu^2}{4} + \frac{\nu}{4} + \frac{\mu^2}{2} + \frac{\mu}{2} + 2i\int\limits_{(0,-i)}^{(s_1,s_2)}\left(a_{0,0}^+ da_{0,0}^- + \sqrt{2}b_{1,1}^+ db_{1,1}^-\right). \tag{4.57}$$

Hence, our task is to evaluate the integral in the right hand side of (4.57). To this end, it is convenient to introduce new monodromy parameters ρ and $\tilde{\eta}$ by the equations

$$(1 + s_1 s_2)^{-1} = e^{i\pi\rho}, \qquad s_3^{-1} = e^{i\pi\tilde{\eta}}. \tag{4.58}$$

The parameters ρ and $\tilde{\eta}$ are determined up to even integer, which will not affect our computation. We transforms the integral in (4.57) into

$$2i\int\limits_{(0,-i)}^{(s_1,s_2)} a^+_{0,0}da^-_{0,0}+\sqrt{2}b^+_{1,1}db^-_{1,1}=-\int\limits_0^{\mu}\mu\left(\ln a^-_{0,0}\right)'_{\mu}d\mu$$
$$-\frac{1}{2}\int\limits_0^{\nu}\nu\left(\ln b^-_{1,1}\right)'_{\nu}d\nu-\frac{i\pi}{2}\int\limits_{(0,-i)}^{(s_1,s_2)}\left(\nu d\rho+2\mu d\tilde{\eta}\right). \tag{4.59}$$

The first two integrals on the right can be rewritten as

$$\int\limits_0^{\mu}\mu\left(\ln a^-_{0,0}\right)'_{\mu}d\mu=-\frac{6\ln 2+i\pi}{4}\mu^2-\int\limits_0^{\mu}\mu\, d\ln\Gamma\left(-\mu\right),$$
$$\int\limits_0^{\nu}\nu\left(\ln b^-_{1,1}\right)'_{\nu}d\nu=\frac{7\ln 2+i\pi}{4}\nu^2-\int\limits_0^{\nu}\nu\, d\ln\Gamma\left(\nu\right).$$

Using (4.32) we get

$$\int\limits_0^{\mu}\mu\left(\ln a^-_{0,0}\right)'_{\mu}d\mu=-\frac{6\ln 2+i\pi}{4}\mu^2+\frac{\mu}{2}+\frac{\mu^2}{2}+\frac{\mu}{2}\ln(2\pi)+\ln G(1-\mu), \tag{4.60a}$$

$$\int\limits_0^{\nu}\nu\left(\ln b^-_{1,1}\right)'_{\nu}d\nu=\frac{7\ln 2+i\pi}{4}\nu^2+\frac{\nu}{2}-\frac{\nu^2}{2}+\frac{\nu}{2}\ln(2\pi)-\ln G(1+\nu). \tag{4.60b}$$

In order to simplify the third integral, we first notice that due to cyclic relation (1.9) satisfied by the Stokes parameters we may write

$$\mu=-\frac{1}{2\pi i}\ln\left(\frac{1-e^{-2\pi i\eta}}{1-e^{i\pi(\nu-\eta)}}\right),\qquad \rho=-\frac{1}{\pi i}\ln\left(\frac{1-e^{2\pi i\nu}}{1-e^{i\pi(\nu-\eta)}}\right), \tag{4.61}$$

where

$$\eta=\tilde{\eta}-\frac{\sigma}{2} \tag{4.62}$$

That means that ρ and μ can be considered as functions of ν and η. Now we can express the third integral in (4.59) in terms of dilogarithms. We have

$$\int\limits_{(0,-i)}^{(s_1,s_2)}\left(\nu d\rho+2\mu d\tilde{\eta}\right)=\int\limits_{(0,-i)}^{(s_1,s_2)}\left(2\mu d\tilde{\eta}-\rho d\nu\right)+\nu\rho$$
$$=\frac{1}{2\pi^2}\left[\mathrm{Li}_2\left(e^{-2i\pi\eta}\right)+\mathrm{Li}_2\left(e^{2\pi i\nu}\right)-2\,\mathrm{Li}_2\left(e^{i\pi(\nu-\eta)}\right)\right]+\nu\rho, \tag{4.63}$$

where $\mathrm{Li}_2(z)$ denotes the dilogarithm function given by integral (4.34). We can check (4.63) by differentiation and using the fact that $s_1 = 0, s_2 = -i$ corresponds to $\mu = \nu = \eta = 0$. To rewrite (4.63) in terms of Barnes G-function we first notice the relations

$$e^{\frac{i\pi}{2}(4\mu-\eta-\nu)} = \frac{\sin\frac{\pi(\eta-\nu)}{2}}{\sin\pi\eta}, \qquad e^{\frac{i\pi}{2}(2\rho+\eta+\nu)} = \frac{\sin\frac{\pi(\eta-\nu)}{2}}{\sin(-\pi\nu)}. \tag{4.64}$$

which can be verified with the help of (4.61). Using the classical formulae (4.38), (4.39) and (4.64) we get

$$\int_{(0,-i)}^{(s_1,s_2)} (\nu d\rho + 2\mu d\tilde{\eta}) = \frac{i}{\pi}\ln\left(\frac{\hat{G}(\eta)}{\hat{G}(\nu)}\right) - \frac{2i}{\pi}\ln\left(\hat{G}\left(\frac{\eta-\nu}{2}\right)\right) - \frac{\eta^2}{4} - \frac{\nu^2}{4} - \frac{\nu\eta}{2} + 2\mu\eta. \tag{4.65}$$

Plugging (4.65) and (4.60) in (4.57) and (4.59) we arrive at

$$\Upsilon = \Upsilon_0 2^{\frac{3}{2}\mu^2-\frac{7\nu^2}{8}}(2\pi)^{-\frac{\mu}{2}-\frac{\nu}{4}}e^{\frac{\pi i}{8}\left(\eta^2+2\mu^2+2\eta\nu-8\mu\eta\right)}\frac{\sqrt{G(1-\nu)\,\hat{G}(\eta)}}{G(1-\mu)\,\hat{G}\left(\frac{\eta-\nu}{2}\right)}, \tag{4.66}$$

The remaining task is to compute Υ_0.

4.2.1 Asymptotic of tau function for $s_1 = 0$ and $s_2 = -i$

Consider the transcendental Hastings-McLeod solution $u_{\mathrm{HM}}(t)$ of (4.43). We remind that it has the following asymptotics on the real axis:

$$u_{\mathrm{HM}}(t) \simeq \begin{cases} \sqrt{\frac{-t}{2}} + O\left((-t)^{-\frac{1}{4}}\, e^{-\frac{2}{3}\sqrt{2}(-t)^{\frac{3}{2}}}\right), & t \to -\infty, \\ \frac{t^{-\frac{1}{4}}e^{-\frac{2}{3}t^{\frac{3}{2}}}}{2\sqrt{\pi}} + O\left(t^{-\frac{7}{4}}e^{-\frac{2}{3}t^{\frac{3}{2}}}\right), & t \to +\infty. \end{cases}$$

Denote by $H_{\mathrm{HM}}(t)$ the corresponding Hamiltonian. Plugging the asymptotics of $u_{\mathrm{HM}}(t)$ into the definition (4.49), one finds that

$$H_{\mathrm{HM}}(t) = \begin{cases} \frac{t^2}{8} - \frac{1}{16t} + O\left((-t)^{-\frac{1}{4}}\, e^{-\frac{2}{3}\sqrt{2}(-t)^{\frac{3}{2}}}\right), & t \to -\infty, \\ O\left(t^{-1}e^{-\frac{4}{3}t^{\frac{3}{2}}}\right), & t \to +\infty. \end{cases}$$

The rapid decay of H_{HM} as $t \to +\infty$ allows to normalize the tau function associated to the Hastings-McLeod solution by setting

$$\tau_{\mathrm{HM}}(t) := \lim_{t_2\to+\infty}\tau(t, t_2, -i, 0) = \exp\left\{\int_t^{+\infty} H_{\mathrm{HM}}(s)ds\right\}. \tag{4.67}$$

Its asymptotics is then given by

$$\tau_{\mathrm{HM}}(t) \simeq \Upsilon_{\mathrm{HM}} \cdot (-t)^{\frac{1}{16}} e^{-\frac{t^3}{24}}, \quad t \to -\infty, \tag{4.68}$$

The coefficient Υ_{HM} represents the finite part of the integral in (4.67) as $t \to -\infty$. It has been evaluated in [34, 4] and the result reads

$$\Upsilon_{\mathrm{HM}} = 2^{-\frac{1}{48}} e^{-\frac{\zeta'(-1)}{2}}, \tag{4.69}$$

where $\zeta(s)$ denotes the Riemann zeta function. Alternatively, Υ_{HM} can be expressed in terms of the Glaisher-Kinkelin constant $A = e^{\frac{1}{12} - \zeta'(-1)}$ or in terms of the special value $G\left(\frac{1}{2}\right) = 2^{\frac{1}{24}} \pi^{-\frac{1}{4}} e^{\frac{3}{2}\zeta'(-1)}$ of the Barnes function.

The Hastings-McLeod solution is associated, via the Riemann-Hilbert correspondence, to the following Stokes data:

$$s_1 = -i, \quad s_2 = 0, \quad s_3 = i.$$

These parameters do not satisfy genericity conditions (4.50), but we can use $\mathbb{Z}_3$-symmetry of the Riemann-Hilbert problem 1.1. More precisely, the solutions of (4.43) verify the periodicity relation

$$u(t; s_1, s_2, s_3) = e^{\frac{2\pi i}{3}} u\left(t e^{\frac{2\pi i}{3}}; s_3, -s_1, -s_2\right),$$

in which we explicitly indicate the dependence of solutions on monodromy. Introducing a "rotated" Hastings-McLeod solution

$$\tilde{u}_{\mathrm{HM}}(t) := e^{\frac{2\pi i}{3}} u_{\mathrm{HM}}\left(t e^{\frac{2\pi i}{3}}; -i, 0, i\right),$$

one may check that $\tilde{u}_{\mathrm{HM}}(t)$ satisfies Painlevé II equation (4.43) and corresponds to the Stokes data

$$s_1 = 0, \quad s_2 = -i, \quad s_3 = -i.$$

These new parameters do satisfy the conditions (4.50) and we used them as reference point in the main part of this chapter. In the above notations, we have $\sigma = 1$ and $\mu = \eta = \nu = 0$, which implies that $a^-_{0,0} = b^-_{1,1} = 0$. One may also rewrite $a^+_{0,0}$, $b^+_{1,1}$ in (4.45), (4.47) as

$$a^+_{0,0} = \frac{2^{3\mu-1} e^{-\frac{3\pi i}{4}} e^{\frac{i\pi\mu}{2}} \Gamma(1-\mu)\, s_3}{\sqrt{\pi}}, \qquad b^+_{1,1} = \frac{2^{-\frac{7\nu}{2}-\frac{7}{4}} e^{-\frac{3\pi i}{4}} e^{-\frac{i\pi\nu}{2}} \Gamma(1+\nu)(1+s_1 s_2)}{\sqrt{\pi}},$$

so that for $\tilde{u}_{\mathrm{HM}}(t) = u(t;0,-i,-i)$ we get

$$a_{0,0}^{+} = \frac{e^{\frac{3\pi i}{4}}}{2\sqrt{\pi}}, \qquad b_{1,1}^{+} = \frac{2^{-\frac{7}{4}} e^{-\frac{3\pi i}{4}}}{\sqrt{\pi}}.$$

The asymptotics of $u_{\mathrm{HM}}(t)$ may be continued inside the sectors $-\frac{\pi}{3} \le \arg t \le 0$, $\frac{2\pi}{3} \le \arg t \le \pi$, see [49]. We record for later use more terms in the relevant asymptotics of $H_{\mathrm{HM}}(t)$ as $|t| \to \infty$:

$$H_{\mathrm{HM}}(t) \simeq \begin{cases} \dfrac{t^2}{8} - \dfrac{1}{16t} - \dfrac{2^{-\frac{3}{4}} e^{-\frac{2}{3}\sqrt{2}(e^{-i\pi}t)^{\frac{3}{2}}}}{4\sqrt{\pi}\,(e^{-i\pi}t)^{\frac{1}{4}}} + \dfrac{e^{-\frac{4}{3}\sqrt{2}\left(e^{-i\pi}t\right)^{\frac{3}{2}}}}{32\pi t} + O\left(|t|^{-\frac{7}{4}}\right), & \arg t \in \left[\frac{2\pi}{3}, \pi\right], \\ \dfrac{e^{-\frac{4}{3}t^{\frac{3}{2}}}}{8\pi t} + O\left(|t|^{-\frac{5}{2}}\right), \quad |t| \to \infty, & \arg t \in \left[-\frac{\pi}{3}, 0\right]. \end{cases} \tag{4.70}$$

Let $\tilde{H}_{\mathrm{HM}}(t)$ denote the Hamiltonian corresponding to the rotated solution $\tilde{u}_{\mathrm{HM}}(t)$. The tau function associated to this solution may be defined as

$$\tilde{\tau}_{\mathrm{HM}}(t) := \lim_{t_1 \to -\infty} \tau(t_1, t, 0, -i) = \exp\left\{ \int_{-\infty}^{t} \tilde{H}_{\mathrm{HM}}(s)\, ds \right\}. \tag{4.71}$$

Its asymptotics contains the so far unknown coefficient Υ_0 (see (4.56)):

$$\tilde{\tau}_{\mathrm{HM}}(t) \simeq \Upsilon_0 t^{-\frac{1}{16}} e^{\frac{t^3}{24}}, \quad t \to +\infty. \tag{4.72}$$

The main idea of our computation of Υ_0 is to relate the integrals (4.67) and (4.71). To this end let us substitute $e^{\frac{2\pi i}{3}} s = y$ into (4.71) and take into account that $\tilde{H}_{\mathrm{HM}}(t) = e^{\frac{2\pi i}{3}} H_{\mathrm{HM}}\left(te^{\frac{2\pi i}{3}}\right)$. This yields

$$\tilde{\tau}_{\mathrm{HM}}(t) = \exp\left\{ \int_{-\infty e^{\frac{2\pi i}{3}}}^{te^{\frac{2\pi i}{3}}} H_{\mathrm{HM}}(y) dy \right\},$$

where the integral is taken along the line $e^{\frac{2\pi i}{3}}\mathbb{R}$.

It follows from (4.68) that

$$\ln \Upsilon_{\mathrm{HM}} = \lim_{t\to+\infty} \left(\ln \tau_{\mathrm{HM}}(-t) - \frac{t^3}{24} - \frac{\ln t}{16} \right) = \lim_{t\to+\infty} \left(\int_{-t}^{+\infty} H_{\mathrm{HM}}(s)\, ds - \frac{t^3}{24} - \frac{\ln t}{16} \right).$$

Since the above integral converges, we may write for $a > 0$

$$\ln \Upsilon_{\mathrm{HM}} = \lim_{t\to+\infty} \left(\int_{-a}^{t} H_{\mathrm{HM}}(s) ds + \int_{-t}^{-a} H_{\mathrm{HM}}(s) ds - \frac{t^3}{24} - \frac{\ln t}{16} \right) =$$
$$= \lim_{t\to+\infty} \left(\int_{-a}^{t} H_{\mathrm{HM}}(s)\, ds + \int_{-t}^{-a} \left[H_{\mathrm{HM}}(s) - \frac{s^2}{8} + \frac{1}{16s} \right] ds - \frac{a^3}{24} - \frac{\ln a}{16} \right).$$

Here the branch cut for the logarithm is chosen to be the negative imaginary axis so that $-\frac{\pi}{2} \le \arg z < \frac{3\pi}{2}$. For $\ln \Upsilon_0$, one similarly obtains from (4.72)

$$\ln \Upsilon_0 = \lim_{t\to+\infty}\left(\ln \tilde{\tau}_{\mathrm{HM}}(t) - \frac{t^3}{24} + \frac{\ln t}{16}\right) = \lim_{t\to+\infty}\left(\int_{-\infty e^{\frac{2\pi i}{3}}}^{te^{\frac{2\pi i}{3}}} H_{\mathrm{HM}}(s)\,ds - \frac{t^3}{24} + \frac{\ln t}{16}\right) =$$

$$= \lim_{t\to+\infty}\left(\int_{-te^{\frac{2\pi i}{3}}}^{ae^{\frac{2\pi i}{3}}} H_{\mathrm{HM}}(s)\,ds + \int_{ae^{\frac{2\pi i}{3}}}^{te^{\frac{2\pi i}{3}}} H_{\mathrm{HM}}(s)\,ds - \frac{t^3}{24} + \frac{\ln t}{16}\right) =$$

$$= \lim_{t\to+\infty}\left(\int_{-te^{\frac{2\pi i}{3}}}^{ae^{\frac{2\pi i}{3}}} H_{\mathrm{HM}}(s)\,ds + \int_{ae^{\frac{2\pi i}{3}}}^{te^{\frac{2\pi i}{3}}} \left[H_{\mathrm{HM}}(s) - \frac{s^2}{8} + \frac{1}{16s}\right] ds - \frac{a^3}{24} + \frac{\ln a}{16}\right).$$

We would like to deform the contours in the two integrals so as to connect $\ln \Upsilon_0$ with $\ln \Upsilon_{\mathrm{HM}}$. The relevant deformations are represented in Figure 4.9.

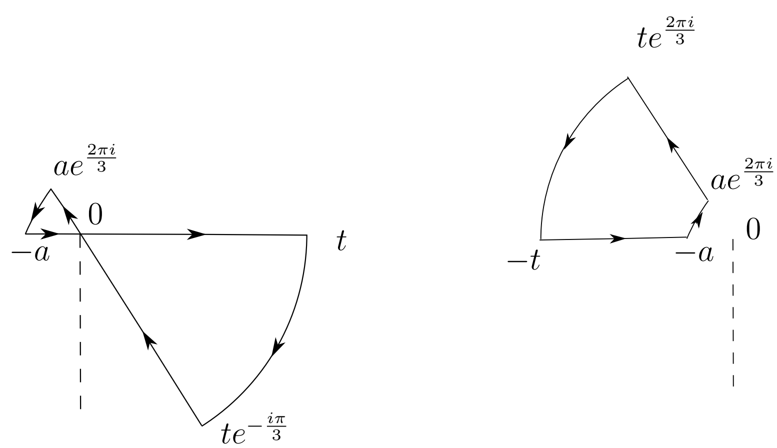

Figure 4.9. Contour deformation for the first (left) and second (right) integral

The crucial observation is that the integrals along the contours shown in Figure 4.9 are equal to zero. The reason for their vanishing is the absense of poles in the Hastings-Mcleod solution inside the sectors $\arg t \in \left[-\frac{\pi}{3}, 0\right] \cup \left[\frac{2\pi}{3}, \pi\right]$, see [65]. It follows that

$$\ln \Upsilon_0 = \lim_{t\to+\infty} \left\{ \int_t^{-a} H_{\text{HM}}(s)\, ds + \int_{-a}^{ae^{\frac{2\pi i}{3}}} H_{\text{HM}}(s)\, ds + \int_{te^{-\frac{\pi i}{3}}}^{t} H_{\text{HM}}(s)\, ds \right.$$
$$+ \int_{-a}^{-t} \left[H_{\text{HM}}(s) - \frac{s^2}{8} + \frac{1}{16s} \right] ds + \int_{ae^{\frac{2\pi i}{3}}}^{-a} \left[H_{\text{HM}}(s) - \frac{s^2}{8} + \frac{1}{16s} \right] ds$$
$$\left. + \int_{-t}^{te^{\frac{2\pi i}{3}}} \left[H_{\text{HM}}(s) - \frac{s^2}{8} + \frac{1}{16s} \right] ds - \frac{a^3}{24} + \frac{\ln a}{16} \right\}.$$

Using the asymptotics (4.70) and appropriate version of the Jordan's lemma, one may show that the limits of the integrals over two arcs of the big circle are equal to zero. Therefore we get

$$\ln \Upsilon_0 = \lim_{t\to+\infty} \left\{ \int_t^{-a} H_{\text{HM}}(s)\, ds + \int_{-a}^{-t} \left[H_{\text{HM}}(s) - \frac{s^2}{8} + \frac{1}{16s} \right] ds + \frac{a^3}{24} + \frac{\ln a}{16} + \frac{i\pi}{48} \right\}$$
$$= -\ln \Upsilon_{\text{HM}} + \frac{i\pi}{48}.$$

In combination with (4.69), this gives us the unknown constant in the connection coefficient (4.66)

$$\Upsilon_0 = 2^{\frac{1}{48}} e^{\frac{\zeta'(-1)}{2} + \frac{i\pi}{48}}.$$

This evaluation reproduces the numerically observed value $\Upsilon_0 \approx 0.865 + 0.114i$ and, together with (4.66) we get the following theorem

Theorem 4.2 ([69]). *Under the genericity assumptions* (4.50) *on the monodromy data, the asymptotic of the homogeneous Painlevé II tau function as* $t_1 \to -\infty$ *and* $t_2 \to +\infty$ *is given by* (4.54) *with*

$$\Upsilon = 2^{\frac{3}{2}\mu^2 - \frac{7\nu^2}{8}} (2\pi)^{-\frac{\mu}{2} - \frac{\nu}{4}} e^{\frac{\pi i}{8}\left(\eta^2 + 2\mu^2 + 2\eta\nu - 8\mu\eta\right)} \frac{\sqrt{G(1-\nu)\, \hat{G}(\eta)}}{G(1-\mu)\, \hat{G}\left(\frac{\eta-\nu}{2}\right)} 2^{\frac{1}{48}} e^{\frac{\zeta'(-1)}{2} + \frac{i\pi}{48}}, \tag{4.73}$$

where μ, ν, σ *and* η *are related to Stokes parameters* s_1, s_2, s_3 *by* (4.45), (4.47), (4.50), (4.58), (4.62), *and* $\hat{G}(z)$ *is the combination of Barnes G-functions given by* (4.40).

Remark 4.3. *If $s_3 = 0$ then parameter η is undefined and we can introduce another parameter λ via $s_1^{-1} = e^{i\pi\tilde{\lambda}}$ and perform similar calculations. Or we can take into account asymptotic formula for the Barnes G-function*

$$\log G(1+z) = \frac{z^2}{2}\ln z - \frac{3z^2}{4} + \frac{z}{2}\ln(2\pi) - \frac{1}{12}\ln z + \frac{1}{12} - \ln A + O\left(\frac{1}{z^2}\right), \quad z \to \infty, \quad |\arg z| < \pi.$$

Then $s_3 = 0$ means $\eta \to -i\infty$, $\mu \to 0$, $\mu\eta \to 0$ and we get

$$\Upsilon = 2^{-\frac{7\nu^2}{8}}(2\pi)^{\frac{\nu}{4}} e^{\frac{i\pi\nu^2}{8}} \sqrt{G\left(1-\nu\right)} 2^{\frac{1}{48}} e^{\frac{\zeta'(-1)}{2} - \frac{i\pi}{48}}.$$

If $1 + s_1 s_2 = 0$ then parameter ρ is undefined but the final answer (4.73) *is holomorphic in η and ν, therefore it stays the same if $1 + s_1 s_2 = 0$.*

Remark 4.4. *It was shown in [69] that the variables $(2\pi i\mu, \eta)$ and $(i\pi\rho, \nu)$ are Darboux coordinates for the symplectic form given by differential $d\omega$ of form* (2.16)*. In fact, one has that*

$$d\omega = 2\pi i d\mu \wedge d\eta = \pi i d\rho \wedge d\nu.$$

That shows Conjecture 2.1 in the setting of system (1.2)*.*

5. HAMILTONIAN AND SYMPLECTIC STRUCTURE OF ISOMONODROMIC DEFORMATIONS

We return to the quotient space $\mathcal{A}_0$ introduced in (2.14). We denote the points $f \in \mathcal{A}_0$ as

$$f = (f_1, \dots, f_{2d}), \quad 2d = \dim \mathcal{M}.$$

Introduce the differential δ on the space $\mathcal{A}_0$. Consider the form

$$\omega_a = \sum_{\nu=1,\dots,n,\infty} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(G^{(\nu)}(z)^{-1} A(z)\, \delta G^{(\nu)}(z)\right). \tag{5.1}$$

Form $\delta\omega_a$ is closed form on $\mathcal{A}_0$. It was shown in [12] that for Fuchsian systems it coincides with the extension of Kirillov-Kostant symplectic structure and is nondegenerate. It also was computed in terms of monodromy data $\mathcal{M}$ using the relation to the form (2.15). Similar identification and proof of non-degeneracy for the case of systems with irregular singularities remains an open problem and we formulate it as a conjecture.

Conjecture 5.1. *Form $\delta\omega_a$ is nondegenerate form on $\mathcal{A}_0$.*

If this conjecture is true, then we have symplectic form $\delta\omega_a$ on $\mathcal{A}_0$. We expect that the isomonodromic deformations become Hamiltonian systems with respect to this form after proper choice of Hamiltonians.

The Hamiltonian and symplectic structure of isomonodromic deformations was studied in [84, 14, 15, 122, 63, 91, 2]. It is expected usually, that Jimbo-Miwa-Ueno tau function is the generating function for the Hamiltonians

$$\frac{\partial \ln \tau\left(\vec{t}, M\right)}{\partial t_k} = H_k\big|_{A\left(z;\vec{t},M\right)}. \tag{5.2}$$

But as we will see in Chapter 6, such statement is sensitive to the choice of symplectic form and Darboux coordinates.

As far as known to the author, there is no explicit description of the Hamiltonian and corresponding symplectic form in general isomonodromic setting. Below we provide the method to construct such Hamiltonian assuming Conjecture 5.1. We start with the formula for the form $\delta\omega_a$

$$\begin{gathered}\delta\omega_a = \sum_{\nu=1,\ldots,n,\infty} \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \delta A \wedge \delta G^{(\nu)}\right) \\ - \sum_{\nu=1,\ldots,n,\infty} \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \delta G^{(\nu)} \left(G^{(\nu)}\right)^{-1} \wedge A\delta G^{(\nu)}\right).\end{gathered} \tag{5.3}$$

Let us assume that there is a family of Hamiltonians $\{H_k\}_{k=1}^{\dim\mathcal{T}}$ with Hamiltonian vector fields X_{H_k} defined by the formula

$$\iota_{X_{H_k}}\delta\omega_a = -\delta H_k, \quad k = 1\ldots\dim\mathcal{T} \tag{5.4}$$

where ι denotes the interior product. The dynamics induced by this Hamiltonians on $\mathcal{A}_0$ is described by

$$\frac{df}{dt_k} = X_{H_k}[f], \quad f \in \mathcal{A}_0, \quad k = 1\ldots\dim\mathcal{T}. \tag{5.5}$$

If we turn it on we will have

$$\delta A[X_{H_k}] = \frac{dA}{dt_k} - \frac{\partial A}{\partial t_k}, \quad \delta G^{(\nu)}[X_{H_k}] = \frac{dG^{(\nu)}}{dt_k} - \frac{\partial G^{(\nu)}}{\partial t_k}, \quad k = 1\ldots\dim\mathcal{T}. \tag{5.6}$$

Here partial derivative means differentiation only of explicit dependence of corresponding functions on times.

Introduce the following form

$$\Omega_k = \sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\frac{\partial A}{\partial t_k}\delta G^{(\nu)}\left(G^{(\nu)}\right)^{-1}\right) - \sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\frac{d(\delta\Theta_\nu)}{dz}\left(G^{(\nu)}\right)^{-1}\frac{\partial G^{(\nu)}}{\partial t_k}\right) \tag{5.7}$$

We rewrite (5.4) in more details and get the following result.

Lemma 5.1. *Assume that dynamics* (5.5) *induced by Hamiltonians* $\{H_k\}_{k=1}^{\dim \mathcal{T}}$ *is isomonodromic and is described by equations* (2.20), (2.10). *Then*

$$\delta H_k = \Omega_k. \tag{5.8}$$

Proof. Taking the interior product of the forrm (5.3) and using (5.6) we get

$$\iota_{X_{H_k}} \delta\omega_a = I_1 + I_2,$$

where

$$\begin{aligned} I_1 = &\sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \frac{dA}{dt_k} \delta G^{(\nu)}\right) - \sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \delta A \frac{dG^{(\nu)}}{dt_k}\right) \\ &- \sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \frac{dG^{(\nu)}}{dt_k} \left(G^{(\nu)}\right)^{-1} A \delta G^{(\nu)}\right) \\ &+ \sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \delta G^{(\nu)} \left(G^{(\nu)}\right)^{-1} A \frac{dG^{(\nu)}}{dt_k}\right), \\ I_2 = -&\sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \frac{\partial A}{\partial t_k} \delta G^{(\nu)}\right) + \sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \delta A \frac{\partial G^{(\nu)}}{\partial t_k}\right) \\ &+ \sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \frac{\partial G^{(\nu)}}{\partial t_k} \left(G^{(\nu)}\right)^{-1} A \delta G^{(\nu)}\right) \\ &- \sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \delta G^{(\nu)} \left(G^{(\nu)}\right)^{-1} A \frac{\partial G^{(\nu)}}{\partial t_k}\right). \end{aligned}$$

Let us work with I_1. We apply (2.10) and get

$$\begin{aligned} I_1 = &\sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \frac{dU_k}{dz} \delta G^{(\nu)}\right) + \sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} [U_k, A] \delta G^{(\nu)}\right) \\ - &\sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \delta A \frac{dG^{(\nu)}}{dt_k}\right) + \sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left[A, \frac{dG^{(\nu)}}{dt_k} \left(G^{(\nu)}\right)^{-1}\right] \delta G^{(\nu)} \left(G^{(\nu)}\right)^{-1}\right). \end{aligned} \tag{5.9}$$

Consider the first term. We replace U_k using (2.20)

$$\begin{aligned} \sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \frac{dU_k}{dz} \delta G^{(\nu)}\right) &= \sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \frac{d}{dz}\left(\frac{dG^{(\nu)}}{dt_k} \left(G^{(\nu)}\right)^{-1}\right) \delta G^{(\nu)}\right) \\ &+ \sum_\nu \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1} \frac{d}{dz}\left(G^{(\nu)} \frac{d\Theta_\nu}{dt_k} \left(G^{(\nu)}\right)^{-1}\right) \delta G^{(\nu)}\right) \end{aligned} \tag{5.10}$$

Since $G^{(\nu)}$ is regular near a_ν the first term on the right hand side is zero. The second term we integrate by parts using (2.22). Taking into account (2.18) we have

$$\begin{aligned}\sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1}\frac{d}{dz}\left(G^{(\nu)}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\right)\delta G^{(\nu)}\right) = \\ \sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1}AG^{(\nu)}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\delta G^{(\nu)}\right) \\ -\sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\frac{d\Theta_\nu}{dz}\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\delta G^{(\nu)}\right) - \sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\frac{d}{dz}(\delta G^{(\nu)})\right).\end{aligned} \tag{5.11}$$

We can use (2.18) again to write

$$\frac{d}{dz}(\delta G^{(\nu)}) = \delta AG^{(\nu)} + A\delta G^{(\nu)} - \delta G^{(\nu)}\frac{d\Theta_\nu}{dz} - G^{(\nu)}\frac{d}{dz}(\delta\Theta_\nu).$$

Therefore we have

$$\begin{aligned}-\sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\frac{d}{dz}(\delta G^{(\nu)})\right) = -\sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\delta AG^{(\nu)}\right) \\ -\sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}A\delta G^{(\nu)}\right) + \sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}\delta G^{(\nu)}\frac{d\Theta_\nu}{dz}\right) \\ +\sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\frac{d\Theta_\nu}{dt_k}\frac{d}{dz}(\delta\Theta_\nu)\right).\end{aligned} \tag{5.12}$$

The last term here has pole at least of order 2 and therefore vanishes.

Now combining (5.10), (5.11), (5.12) we can write

$$\begin{aligned}\sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\left(G^{(\nu)}\right)^{-1}\frac{dU_k}{dz}\delta G^{(\nu)}\right) = -\sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\delta\left(\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}AG^{(\nu)}\right)\right) \\ +\sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\frac{d}{dt_k}(\delta\Theta_\nu)\left(G^{(\nu)}\right)^{-1}AG^{(\nu)}\right).\end{aligned} \tag{5.13}$$

We use (2.20) in the first term on the right hand side and get

$$\begin{aligned}-\sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\delta\left(\frac{d\Theta_\nu}{dt_k}\left(G^{(\nu)}\right)^{-1}AG^{(\nu)}\right)\right) = -\sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\delta\left(U_kA\right)\right) \\ +\sum_\nu \mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\delta\left(\frac{dG^{(\nu)}}{dt_k}\left(G^{(\nu)}\right)^{-1}A\right)\right).\end{aligned} \tag{5.14}$$

The first term here is zero, since δU_kA is the rational function and sum of its residues is zero. To simplify second term we take δ of (2.20). We get

$$\delta\left(\frac{dG^{(\nu)}}{dt_k}\right) = \delta U_kG^{(\nu)} + U_k\delta G^{(\nu)} - \delta G^{(\nu)}\frac{d\Theta_\nu}{dt_k} - G^{(\nu)}\delta\left(\frac{d\Theta_\nu}{dt_k}\right).$$

Replacing here $\dfrac{d\Theta_\nu}{dt_k}$ in the third term according to (2.20) we get

$$\delta\left(\frac{dG^{(\nu)}}{dt_k}\right)=\delta U_k G^{(\nu)}+U_k\delta G^{(\nu)}-\delta G^{(\nu)}\left(G^{(\nu)}\right)^{-1}U_k G^{(\nu)}+\delta G^{(\nu)}\left(G^{(\nu)}\right)^{-1}\frac{dG^{(\nu)}}{dt_k}-G^{(\nu)}\delta\left(\frac{d\Theta_\nu}{dt_k}\right).$$

Using this formula, (5.13) and (5.14) we have

$$\sum_\nu \operatorname{res}_{z=a_\nu}\operatorname{Tr}\left(\left(G^{(\nu)}\right)^{-1}\frac{dU_k}{dz}\delta G^{(\nu)}\right)=\sum_\nu \operatorname{res}_{z=a_\nu}\operatorname{Tr}\left(\frac{dG^{(\nu)}}{dt_k}\left(G^{(\nu)}\right)^{-1}\delta A\right)$$
$$+\sum_\nu \operatorname{res}_{z=a_\nu}\operatorname{Tr}\left(A\,\delta U_k\right)+\sum_\nu \operatorname{res}_{z=a_\nu}\operatorname{Tr}\left([A,U_k]\delta G^{(\nu)}\left(G^{(\nu)}\right)^{-1}\right)$$
$$+\sum_\nu \operatorname{res}_{z=a_\nu}\operatorname{Tr}\left(\left[\frac{dG^{(\nu)}}{dt_k}\left(G^{(\nu)}\right)^{-1},A\right]\delta G^{(\nu)}\left(G^{(\nu)}\right)^{-1}\right).$$

The second term here is zero since $A\,\delta U_k$ is rational function on z. Plugging this formula in (5.9) we get $I_1=0$.

Let us work now with I_2. Using (2.18) we can write

$$\begin{aligned}\delta A=\delta G^{(\nu)}\frac{d\Theta_\nu}{dz}\left(G^{(\nu)}\right)^{-1}-G^{(\nu)}\frac{d\Theta_\nu}{dz}\left(G^{(\nu)}\right)^{-1}\delta G^{(\nu)}\left(G^{(\nu)}\right)^{-1}\\ +G^{(\nu)}\frac{d}{dz}(\delta\Theta_\nu)\left(G^{(\nu)}\right)^{-1}+\delta\left(\frac{dG^{(\nu)}}{dz}\left(G^{(\nu)}\right)^{-1}\right).\end{aligned}\tag{5.15}$$

Using (2.18) and (5.15) we get rid of A and δA in I_2. We have

$$I_2=-\sum_\nu \operatorname{res}_{z=a_\nu}\operatorname{Tr}\left(\left(G^{(\nu)}\right)^{-1}\frac{\partial A}{\partial t_k}\delta G^{(\nu)}\right)+\sum_\nu \operatorname{res}_{z=a_\nu}\operatorname{Tr}\left(\frac{d(\delta\Theta_\nu)}{dz}\left(G^{(\nu)}\right)^{-1}\frac{\partial G^{(\nu)}}{\partial t_k}\right)$$
$$+\sum_\nu \operatorname{res}_{z=a_\nu}\operatorname{Tr}\left(\delta\left(\frac{dG^{(\nu)}}{dz}\left(G^{(\nu)}\right)^{-1}\right)\frac{\partial G^{(\nu)}}{\partial t_k}\left(G^{(\nu)}\right)^{-1}\right)\quad.$$
$$+\sum_\nu \operatorname{res}_{z=a_\nu}\operatorname{Tr}\left(\frac{dG^{(\nu)}}{dz}\left(G^{(\nu)}\right)^{-1}\left[\delta G^{(\nu)}\left(G^{(\nu)}\right)^{-1},\frac{\partial G^{(\nu)}}{\partial t_k}\left(G^{(\nu)}\right)^{-1}\right]\right)$$

The last two terms here are zero, since $G^{(\nu)}$ is regular near a_ν. Using (5.4) we now get (5.8).

□

This result allows us to formulate the conjecture

Conjecture 5.2. *Form Ω_k given by* (5.7) *is exact.*

If this conjecture holds, it provides the formula for Hamiltonians in general case. Below in Chapter 6 we check Conjecture 5.2 and Conjecture 5.1 for Painlevé equations.

6. ISOMONODROMIC SETTING OF PAINLEVÉ EQUATIONS

6.1 PI equation

The linear system associated with the first Painlevé equation is the 2×2 matrix ODE with one irregular singular point of Poincaré rank 5 at $z = \infty$ and with one Fuchsian singular point at $z = 0$,

$$\frac{d\Phi}{dz} = A(z)\,\Phi, \qquad A(z) = A_{\infty,-5} z^4 + A_{\infty,-3} z^2 + A_{\infty,-2} z + A_{\infty,-1} + \frac{A_{0,0}}{z}. \tag{6.1}$$

The matrix coefficients are,

$$A_{\infty,-5} = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}, \quad A_{\infty,-3} = \begin{pmatrix} 0 & -4u \\ 4u & 0 \end{pmatrix}, \quad A_{\infty,-2} = \begin{pmatrix} 0 & -v \\ -v & 0 \end{pmatrix},$$

$$A_{\infty,-1} = \begin{pmatrix} 2u^2 + x & -2u^2 - x \\ 2u^2 + x & -2u^2 - x \end{pmatrix}, \quad A_{0,0} = -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and so that the space $\mathcal{A}$ is parametrized by v, u, and t,

$$\mathcal{A} = \{(v, u, x)\}.$$

It is not generic system with one irregular singular point of Poincaré rank 5 at $z = \infty$ and with one Fuchsian singular point at $z = 0$ and dimension of $\mathcal{A}$ is not given by the formula (2.2).

The formal solution at $z = \infty$ is given by the series,

$$\Phi_{\text{form}}(z) = \left(I + \frac{g_1}{z} + \frac{g_2}{z^2} + \frac{g_3}{z^3} + \frac{g_4}{z^4} + \frac{g_5}{z^5} + O\left(\frac{1}{z^6}\right) \right) e^{\Theta(z)}, \quad \Theta(z) = \sigma_3 \left(\frac{4z^5}{5} + xz \right) \tag{6.2}$$

with the explicit formulae for the first five coefficient matrices g_k given by the equations,

$$g_1 = \frac{1}{2} \begin{pmatrix} -H & 0 \\ 0 & H \end{pmatrix}, \quad g_2 = \begin{pmatrix} \frac{H^2}{8} & \frac{u}{2} \\ \frac{u}{2} & \frac{H^2}{8} \end{pmatrix}, \quad g_3 = \begin{pmatrix} -\frac{H^3}{48} - \frac{v - x^2}{24} & \frac{uH}{4} + \frac{v}{8} \\ -\frac{uH}{4} - \frac{v}{8} & \frac{H^3}{48} + \frac{v - x^2}{24} \end{pmatrix}, \tag{6.3}$$

$$g_4 = \begin{pmatrix} \frac{H^4}{384} + \frac{v-x^2}{48}H + \frac{u^2}{8} & \frac{uH^2}{16} + \frac{vH}{16} + \frac{2u^2+x}{8} \\ \frac{uH^2}{16} + \frac{vH}{16} + \frac{2u^2+x}{8} & \frac{H^4}{384} + \frac{v-x^2}{48}H + \frac{u^2}{8} \end{pmatrix}, \tag{6.4}$$

$$g_5 = \begin{pmatrix} -\frac{H^5}{3840} - \frac{v-x^2}{192}H^2 - \frac{5u^2-2x}{80}H - \frac{2vu+1}{160} & \frac{uH^3}{96} + \frac{vH^2}{64} + \frac{2u^2+x}{16}H + \frac{v-x^2}{48}u + \frac{1}{16} \\ -\frac{uH^3}{96} - \frac{vH^2}{64} - \frac{2u^2+x}{16}H - \frac{v-x^2}{48}u - \frac{1}{16} & \frac{H^5}{3840} + \frac{v-x^2}{192}H^2 + \frac{5u^2-2x}{80}H + \frac{2vu+1}{160} \end{pmatrix}. \tag{6.5}$$

where

$$H = \frac{v^2}{4} - 4u^3 - 2xu. \tag{6.6}$$

The Fuchsian point $z = 0$ is a resonant point and hence the generic theory outlined in the Section 2.1 is not applicable. In fact, the behavior of the solution $\Phi(z)$ at $z = 0$ is given by the formula [87],

$$\Phi(z) = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} z^{-\frac{1}{2}\sigma_3}\hat{\Phi}(z),$$

where $\hat{\Phi}(z)$ is holomorphic and invertible at $z = 0$.

The isomonodromic deformation of system (6.14) with respect to parameter t yields the linear matrix differential equation

$$\begin{gathered} \frac{d\Phi}{dx} = U(z)\,\Phi, \qquad B(z) = U_1 z + \frac{U_{-1}}{z}, \\ U_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_{-1} = \begin{pmatrix} u & -u \\ u & -u \end{pmatrix}. \end{gathered} \tag{6.7}$$

The compatibility condition of (6.1) and (6.7) yields the system of ODEs

$$\begin{aligned} \frac{du}{dx} &= \frac{v}{2}, \\ \frac{dv}{dx} &= 12u^2 + 2x. \end{aligned} \tag{6.8}$$

which is equivalent to the PI equation.

We are passing now to the forms ω_{JMU} and ω corresponding to the Painlevé I system (6.1). Because $z = 0$ is the resonant Fuchsian point we strictly speaking can not use the definitions (2.12) and (2.16) for the forms ω_{JMU} and ω. However, following [87], we take (2.12) and (2.16), where the undefined contribution of the resonant point $z = 0$ simply ignored, as the definitions of these forms in the case of the Painlevé I equations, i.e. we put

$$\omega_{\text{JMU}} = -\operatorname{res}_{z=\infty} \operatorname{Tr}\left(G^{-1}(z)\frac{dG(z)}{dz}\frac{d\Theta(z)}{dt}\right)dt, \tag{6.9}$$

and

$$\omega = \mathrm{res}_{z=\infty} \operatorname{Tr}\left(A(z)\, dG(z)\, G(z)^{-1}\right), \tag{6.10}$$

where $G(z)$ and $\Theta(z)$ are the series and the exponent from (6.2). We obtain from (6.9) at once that

$$\omega_{\mathrm{JMU}} \equiv \frac{d\ln\tau}{dx}dx = Hdx.$$

The form ω needs more work.

Introduce the matrix coefficients h_j of the series inverse to the series (6.2),

$$\left(I + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \frac{h_4}{z^4} + O\left(\frac{1}{z^5}\right)\right) = \left(I + \frac{g_1}{z} + \frac{g_2}{z^2} + \frac{g_3}{z^3} + \frac{g_4}{z^4} + \frac{g_5}{z^5} + O\left(\frac{1}{z^6}\right)\right)^{-1}. \tag{6.11}$$

We have for the first four coefficients the relations,

$$h_1 = -g_1, \quad h_2 = -g_2 + g_1^2, \quad h_3 = -g_3 + g_2g_1 + g_1g_2 - g_1^3,$$

$$h_4 = -g_4 + g_3g_1 + g_1g_3 + g_2^2 - g_2g_1^2 - g_1g_2g_1 - g_1^2g_2 + g_1^4.$$

Plugging (6.11) and (6.2) into (6.10) we arrive at the formula,

$$\omega = \operatorname{Tr}\Big(-A_4(h_4dg_1 + h_3dg_2 + h_2dg_3 + h_1dg_4 + dg_5) - A_2(h_2dg_1 + h_1dg_2 + dg_3)$$
$$-A_1(h_1dg_1 + dg_2) - A_0dg_1\Big).$$

Using (6.3)–(6.5) we get after (quite a lot of) simplifications

$$\omega = \frac{3}{5}vdu - \frac{2}{5}udv - \frac{1}{5}Hdx + \frac{4}{5}xdH,$$

and properly combining the terms,

$$\omega = \left[vdu - Hdx + d\left(\frac{4Hx}{5} - \frac{2vu}{5}\right)\right]. \tag{6.12}$$

Equation (6.12) proves Conjecture 2.2, in the case of the 2×2 system (6.1) and gives the explicit formula for $G(v, u, x)$,

$$G(v, u, x) = \frac{1}{5}\Big(4Hx - 2vu\Big).$$

The corresponding equation (2.32) is

$$\frac{d\ln\tau}{dx} = \left(v\frac{du}{dx} - H\right) + \frac{1}{5}\frac{d}{dx}\Big(4Hx - 2vu\Big),$$

and, of course, can be easily checked directly.

Let us show now the Conjecture 5.1. Plugging (6.11) and (6.2) into (5.1) we arrive at the formula,

$$\omega_a = \mathrm{Tr}\Big(-A_{\infty,-5}(h_4\delta g_1 + h_3\delta g_2 + h_2\delta g_3 + h_1\delta g_4 + \delta g_5) - A_{\infty,-3}(h_2\delta g_1 + h_1\delta g_2 + \delta g_3)$$
$$-A_{\infty,-2}(h_1\delta g_1 + \delta g_2) - A_{\infty,-1}\delta g_1\Big).$$

Using (6.3)–(6.5) we get again

$$\omega_a = \frac{3}{5}v\delta u - \frac{2}{5}u\delta v + \frac{4}{5}x\delta H.$$

After taking differential δ it becomes nondegenerate 2-form

$$\delta\omega_a = \delta v \wedge \delta u. \tag{6.13}$$

That shows Conjecture 5.1.

For the form (5.7) we have

$$\Omega = \mathrm{Tr}\Big(-\frac{\partial A_{\infty,-1}}{\partial x}\delta g_1\Big) = \delta H,$$

with H given by (6.6). That means that Conjecture 5.2 holds. We can easily check that system (6.8) is Hamiltonian system with symplectic form (6.13) and the Hamiltonian (6.6).

6.2 PII equation

According to [76], the second Painlevé equation describes the isomonodromic deformations of the 2×2 linear system having only one irregular singular point at $z = \infty$ of the Poincaré rank 3.

$$\frac{d\Phi}{dz} = A(z)\,\Phi, \qquad A(z) = A_{-3}z^2 + A_{-2}z + A_{-1}.$$
$$A_{-3} = \sigma_3, \quad A_{-2} = \begin{pmatrix} 0 & k \\ -\frac{2v}{k} & 0 \end{pmatrix}, \quad A_{-1} = \begin{pmatrix} v + \frac{x}{2} & -ku \\ -\frac{2}{k}(\theta + vu) & -v - \frac{x}{2} \end{pmatrix}. \tag{6.14}$$

This representation is differs from the one considered in the introduction (see [48]). The complex parameters v, u, k, θ, and x can be taken as the original coordinates on the corresponding space $\mathcal{A}$. Its dimension is given by (2.2).

$$\mathcal{A} = \{(v, u, k, \theta, x)\}.$$

The formal solution of system (6.14) at its only (irregular) singular point, $z=\infty$, has the structure

$$\Phi_{\text{form}}(z) \simeq \left(I + \sum_{m=1}^{\infty} g_m z^{-m}\right) \exp\left(\sigma_3 \left(\frac{z^3}{3} + \frac{xz}{2} - \theta \ln z\right)\right), \quad z \to \infty, \tag{6.15}$$

with the first three matrix coefficients g_k, $k=1,2,3$ given by the explicit formulae,

$$g_1 = \begin{pmatrix} -H & -\frac{k}{2} \\ -\frac{v}{k} & H \end{pmatrix}, \tag{6.16}$$

$$g_2 = \begin{pmatrix} \frac{H^2}{2} + \frac{v}{4} - \frac{x\theta}{4} & -\frac{kH}{2} + \frac{ku}{2} \\ \frac{vH}{k} - \frac{vu}{k} - \frac{\theta}{k} & \frac{H^2}{2} + \frac{v}{4} + \frac{x\theta}{4} \end{pmatrix}, \tag{6.17}$$

$$g_3 = \begin{pmatrix} -\frac{H^3}{6} - \frac{Hv}{4} + \frac{Hx}{6} + \frac{Hx\theta}{4} + \frac{vu}{6} + \frac{\theta^2}{6} + \frac{\theta}{3} & -\frac{kH^2}{4} + \frac{kuH}{2} + \frac{kv}{8} + \frac{kx}{4} - \frac{kx\theta}{8} \\ -\frac{vH^2}{2k} + \frac{Hvu}{k} + \frac{H\theta}{k} + \frac{v^2}{4k} + \frac{vx\theta}{4k} + \frac{vx}{2k} & \frac{H^3}{6} + \frac{Hv}{4} - \frac{Hx}{6} + \frac{Hx\theta}{4} - \frac{vu}{6} - \frac{\theta^2}{6} + \frac{\theta}{6} \end{pmatrix}, \tag{6.18}$$

where

$$H = \frac{v^2}{2} + vu^2 + \frac{vx}{2} + u\theta. \tag{6.19}$$

The isomonodromic deformation of system (6.14) with respect to parameter t yields the linear matrix differential equation

$$\begin{aligned} &\frac{d\Phi}{dx} = U(z)\,\Phi, \qquad U(z) = U_1 z + U_0, \\ &U_1 = \frac{1}{2}\sigma_3, \quad U_0 = \frac{1}{2}\begin{pmatrix} 0 & k \\ -\frac{2v}{k} & 0 \end{pmatrix}. \end{aligned} \tag{6.20}$$

The compatibility condition of (6.20) and (6.14) reads

$$\begin{aligned} \frac{du}{dx} &= v + u^2 + \frac{x}{2}, \\ \frac{dv}{dx} &= -2vu - \theta, \\ \frac{dk}{dx} &= -ku, \\ \frac{d\theta}{dx} &= 0. \end{aligned} \tag{6.21}$$

The last equation of this system is just the statement that θ, as the part of the monodromy data, is constant. The third equation gives $\ln k(x)$ as the antiderivative of $-u(x)$. The first two first order differential equations are equivalent to one second order differential equation, indeed, the PII equation, with $\alpha = \frac{1}{2} - \theta$.

Let us now discuss the forms $\omega_{\rm JMU}$ and ω corresponding to (6.14). The linear system (6.14) has only ∞ as its singular point. Therefore, the general definition (2.12) of the form $\omega_{\rm JMU}$ transforms to the equation,

$$\omega_{\rm JMU} = -\operatorname{res}_{z=\infty} \operatorname{Tr}\left(G^{-1}(z)\frac{dG(z)}{dz}\frac{d\Theta(z)}{dx}\right) dx.$$

Plugging (6.15) at the right hand side we arrive at the formulae,

$$\omega_{\rm JMU} = -\operatorname{Tr}\left(\frac{1}{2}g_1\sigma_3\right),$$

or, taking into account (6.16) (cf. (5.2),

$$\omega_{\rm JMU} \equiv \frac{d\ln\tau}{dx}dx = H dx,$$

Similarly, the general definition (2.16) of the form ω transforms to the equation,

$$\omega = \operatorname{res}_{z=\infty} \operatorname{Tr}\left(A\left(z\right)dG\left(z\right)G\left(z\right)^{-1}\right). \tag{6.22}$$

Plugging (6.15) into (6.22) we arrive at the formula,

$$\omega = \operatorname{Tr}\Big(A_2 dg_3 - A_2 dg_2 g_1 - A_2 dg_1 g_2 + A_2 dg_1 g_1^2 + A_1 dg_2 - A_1 dg_1 g_1 + A_0 dg_1\Big).$$

Now, it is more involved to plug (6.16) - (6.18) into the right hand side of the last equation. However, after performing some algebra, the final expression comes out rather simple,

$$\omega = -\frac{1}{3}udv + \frac{2}{3}vdu - \theta\frac{dk}{k} + \frac{2}{3}xdH - \frac{1}{3}Hdx - \frac{2\theta-1}{3}d\theta,$$

and can be in turn easily transformed to the equation,

$$\omega = vdu - Hdx + d\left(\frac{2}{3}Hx - \frac{1}{3}vu - \theta\ln k - \frac{\theta^2}{3} + \frac{\theta}{3}\right) + \ln k\, d\theta$$

or, using the definition of the canonical coordinates,

$$p_1 = v, \quad q_1 = u, \quad p_2 = \ln k, \quad q_2 = \theta,$$

we can write

$$\omega = p_1 dq_1 + p_2 dq_2 - Hdx + d\left(\frac{2}{3}Hx - \frac{1}{3}p_1 q_1 - p_2 q_2 - \frac{q_2^2}{3} + \frac{q_2}{3}\right). \tag{6.23}$$

Equation (6.23) proves Conjecture 2.2 in the case of the 2×2 system (6.14). Indeed, this is exactly the formula (2.31) with the specification

$$G(p_1, q_1, p_2, q_2, x) = \frac{2}{3}Hx - \frac{1}{3}p_1 q_1 - p_2 q_2 - \frac{q_2^2}{3} + \frac{q_2}{3}.$$

The corresponding equation (2.32) is

$$\begin{aligned}\frac{d\ln\tau}{dx} &= v\frac{du}{dx} - H + \frac{d}{dx}\left(\frac{2}{3}Hx - \frac{1}{3}uv - \theta\ln k\right)\\ &= v\frac{du}{dx} - H + \frac{d}{dx}\left(\frac{2}{3}Hx - \frac{1}{3}uv\right) + \theta q.\end{aligned}$$

We want to interpret (6.21) as Hamiltonian system using Conjectures 5.1, 5.2. Plugging (6.15) into (5.1) we arrive at the formula,

$$\omega_a = \mathrm{Tr}\Big(A_{-3}\delta g_3 - A_{-3}\delta g_2 g_1 - A_{-3}\delta g_1 g_2 + A_{-3}\delta g_1 g_1^2 + A_{-2}\delta g_2 - A_{-2}\delta g_1 g_1 + A_{-1}\delta g_1\Big).$$

Now, it is more involved to plug (6.16) - (6.18) into the right hand side of the last equation. However, after performing some algebra, the final expression comes out rather simple,

$$\omega_a = -\frac{1}{3}u\delta v + \frac{2}{3}v\delta u - \theta\frac{\delta k}{k} + \frac{2}{3}x\delta H - \frac{2\theta - 1}{3}\delta\theta,$$

We take differential δ and we get

$$\delta\omega_a = \delta v \wedge \delta u + \delta(\ln k) \wedge \delta\theta. \tag{6.24}$$

Therefore Conjecture 5.1 holds. Now for the form (5.7) we have

$$\Omega = \mathrm{Tr}\Big(-\frac{\partial A_{-1}}{\partial x}\delta g_1\Big) = \delta H$$

with H given by (6.19). That means that Conjecture 5.2 holds. We can easily check that system (6.21) is Hamiltonian system with symplectic form (6.24) and the Hamiltonian (6.19).

6.3 PIII(D6) equation

The linear system associated with the third Painlevé equation we take again from [76]. This is the system

$$\frac{d\Phi}{dz} = A(z)\,\Phi, \qquad A(z) = \frac{A_{0,-1}}{z^2} + \frac{A_{0,0}}{z} + A_{\infty,-1}, \tag{6.25}$$

with

$$A_{\infty,-1} = \frac{1}{2}\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, \quad A_{0,0} = \begin{pmatrix} -\theta_\infty & -ukx \\ \frac{vu(x-v)}{kx} + \frac{\theta_0+\theta_\infty}{k} - \frac{2\theta_\infty v}{kx} & \theta_\infty \end{pmatrix},$$

$$A_{0,-1} = \begin{pmatrix} v - \frac{x}{2} & -kx \\ \frac{v(v-x)}{kx} & -v + \frac{x}{2} \end{pmatrix}.$$

The system has two irregular singular points at $z = \infty$ and $z = 0$, both of the Poincaré rank 1.

The corresponding formal solutions are:

$$\Phi^{(\infty)}_{\text{form}}(z) = \left(I + \frac{g_{\infty,1}}{z} + O\left(\frac{1}{z^2}\right)\right) e^{\Theta_\infty(z)}, \quad \Theta_\infty(z) = \sigma_3\left(\frac{xz}{2} - \theta_\infty \ln z\right), \tag{6.26}$$

with

$$g_{\infty,1} = \begin{pmatrix} -\frac{H}{2} - \frac{vu}{2x} + \frac{\theta_\infty^2-\theta_0^2}{2x} + \frac{x}{2} & ku \\ \frac{vu(x-v)}{kx^2} + \frac{\theta_0+\theta_\infty}{kx} - \frac{2\theta_\infty v}{kx^2} & \frac{H}{2} + \frac{vu}{2x} - \frac{\theta_\infty^2-\theta_0^2}{2x} - \frac{x}{2} \end{pmatrix}, \tag{6.27}$$

at $z = \infty$, and

$$\Phi^{(0)}_{\text{form}}(z) = G_0\left(I + g_{0,1}z + O\left(z^2\right)\right) e^{\Theta_0(z)}, \quad \Theta_0(z) = \sigma_3\left(\frac{x}{2z} + \theta_0 \ln z\right) \tag{6.28}$$

with

$$g_{0,1} = \begin{pmatrix} -\frac{H}{2} - \frac{vu}{2x} - \frac{\theta_\infty^2-\theta_0^2}{2x} + \frac{x}{2} & \frac{ua}{x}(v-x) + \frac{a}{x}(\theta_\infty - \theta_0) \\ -\frac{1}{xa}(vu + \theta_0 + \theta_\infty) & \frac{H}{2} + \frac{vu}{2x} + \frac{\theta_\infty^2-\theta_0^2}{2x} - \frac{x}{2} \end{pmatrix}, \tag{6.29}$$

at $z = 0$. In (6.27) and (6.29),

$$H = \frac{1}{x}\left(2v^2u^2 + v(2x - 2xu^2 + (4\theta_\infty - 1)u) - 2xu(\theta_0 + \theta_\infty) + \theta_\infty^2 - \theta_0^2\right), \tag{6.30}$$

and G_0 diagonalizes matrix $A_{0,-1}$,

$$G_0^{-1}A_{0,-1}G_0 = -\frac{x\sigma_3}{2},$$

and it is chosen in the form

$$G_0 = \frac{1}{\sqrt{k}} \begin{pmatrix} k & -k \\ \frac{v}{x} & \frac{x-v}{x} \end{pmatrix} a^{-\frac{\sigma_3}{2}}, \tag{6.31}$$

with a being an extra gauge parameter, so that the full space $\mathcal{A}$ is seven dimensional,

$$\mathcal{A} = \{v, u, k, a, x, \theta_0, \theta_\infty\}, \tag{6.32}$$

as given by the formula (2.2).

From the series (6.26) and (6.28) it follows that θ_∞ and θ_0 are the formal monodromy exponents and the parameter x is the isomonodromic time. The isomonodromicity with respect to x yields the second differential equation for $\Phi(z) \equiv \Phi(z, x)$,

$$\frac{d\Phi}{dx} = U(z)\,\Phi, \qquad U(z) = U_1 z + U_0 + \frac{U_{-1}}{z}, \tag{6.33}$$

where,

$$U_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_0 = \frac{1}{x} \begin{pmatrix} 0 & -ukx \\ \frac{vu(x-v)}{kx} + \frac{\theta_0+\theta_\infty}{k} - \frac{2\theta_\infty v}{kx} & 0 \end{pmatrix},$$

$$U_{-1} = \begin{pmatrix} \frac{x-2v}{2x} & k \\ \frac{v(x-v)}{kx^2} & \frac{2v-x}{2x} \end{pmatrix}.$$

The compatibility condition of the matrix equations (6.25) and (6.33) implies the following dynamical system on (6.32),

$$\begin{gathered} \frac{du}{dx} = \frac{4vu^2}{x} - 2u^2 + \frac{u(4\theta_\infty - 1)}{x} + 2, \\ \frac{dv}{dx} = -\frac{4v^2u}{x} + \frac{v(4xu - 4\theta_\infty + 1)}{x} + 2\theta_0 + 2\theta_\infty, \\ \frac{dk}{dx} = -\frac{4vuk}{x} + 2uk - \frac{2\theta_\infty k}{x}, \qquad \frac{da}{dx} = \frac{a}{x}\left(2ux + 2\theta_0\right), \\ \frac{d\theta_\infty}{dx} = 0, \quad \frac{d\theta_0}{dx} = 0. \end{gathered} \tag{6.34}$$

It should be also mentioned, that the fourth equation, i.e. the equation for the function $a(x)$, follows from plugging (6.28) into equation (6.33) – the second equation of the Lax pair, and equating the terms of zero order with respect to z.

The last two equations of system (6.34) just state that θ_∞ and θ_0 as the part of the monodromy data, are constant. The third and the fourth equations give $\ln k(x)$ and $\ln a(x)$ as the antiderivatives of the simple combinations of v and u. The first two equations are equivalent to the PIII(D6) equation, with

$$\alpha = 8\theta_0, \quad \beta = 4 - 8\theta_\infty, \quad \gamma = 4, \quad \delta = -4.$$

The general formulae (2.12) and (2.16) transform, in the case of system (6.25), into the equations,

$$\begin{aligned} \omega_{\mathrm{JMU}} = &- \mathrm{res}_{z=\infty} \operatorname{Tr}\left(\left(G^{(\infty)}(z)\right)^{-1} \frac{dG^{(\infty)}(z)}{dz} \frac{d\Theta_\infty(z)}{dx}\right) dx \\ &- \mathrm{res}_{z=0} \operatorname{Tr}\left(\left(G^{(0)}(z)\right)^{-1} \frac{dG^{(0)}(z)}{dz} \frac{d\Theta_0(z)}{dx}\right) dx \end{aligned} \tag{6.35}$$

and

$$\omega = \mathrm{res}_{z=\infty} \operatorname{Tr}\left(A(z)\, dG^{(\infty)}(z)\, G^{(\infty)}(z)^{-1}\right) + \mathrm{res}_{z=0} \operatorname{Tr}\left(A(z)\, dG^{(0)}(z)\, G^{(0)}(z)^{-1}\right), \tag{6.36}$$

respectively. Substituting the series $G^{(\infty,0)}(z)$ and the exponentials $\Theta_{\infty,0}(z)$ from (6.26) and (6.28) into (6.35), we obtain that

$$\omega_{\mathrm{JMU}} = -\frac{1}{2}\operatorname{Tr}\Big(g_{\infty,1}\sigma_3\Big)dt - \frac{1}{2}\operatorname{Tr}\Big(g_{0,1}\sigma_3\Big)dt$$

and using (6.27) and (6.29) we arrive at the final formula for ω_{JMU},

$$\omega_{\mathrm{JMU}} \equiv \frac{d\ln\tau}{dx}dx = Hdx + \frac{uv}{x}dx - xdx. \tag{6.37}$$

Note the additional to Hdx terms in the right hand side of (6.37).

Similar substitution of $G^{(\infty,0)}(z)$ from (6.26) and (6.28) into (6.36) leads us to the formula,

$$\omega = \operatorname{Tr}\Big(A_{-1}dG_0G_0^{-1} + G_0^{-1}A_{-2}G_0dg_{0,1} - A_0dg_{\infty,1}\Big),$$

and using again (6.27), (6.29) and (6.31) we arrive at the equation,

$$\omega = vdu + xdH - \theta_\infty\frac{dk}{k} - \theta_0\frac{da}{a} - xdx.$$

After regrouping the terms we obtain that

$$\omega = vdu - Hdx + d\left(Hx - \theta_\infty \ln k - \theta_0 \ln a - \frac{x^2}{2}\right) + \ln k\, d\theta_\infty + \ln a \;\, d\theta_0,$$

or, using the definition of the canonical coordinates,

$$p_1 = v, \quad q_1 = u, \quad p_2 = \ln k, \quad q_2 = \theta_\infty, \quad p_3 = \ln a, \quad q_3 = \theta_0,$$

we can write

$$\omega = p_1 dq_1 + p_2 dq_2 + p_3 dq_3 - Hdt + d\left(Ht - q_2 p_2 - q_3 p_3 - \frac{t^2}{2}\right). \tag{6.38}$$

Equation (6.38) proves Conjecture 2.2, in the case of the 2×2 system (6.25) and gives the explicit formula for $G(p_j, q_j, t)$,

$$G(p_1, p_2, p_3, q_1, q_2, q_3, x) = Hx - q_2 p_2 - q_3 p_3 - \frac{x^2}{2}.$$

The corresponding equation (2.32) is

$$\frac{d \ln \tau}{dx} = v\frac{du}{dx} - H + \frac{d}{dx}\left(Hx - \theta_\infty \ln k - \theta_0 \ln a - \frac{x^2}{2}\right). \tag{6.39}$$

Remark 6.1. *One can deduce from* (6.34) *that*

$$\frac{vu}{x} = \frac{1}{4}\frac{d}{dx}\ln\frac{a}{k} - \frac{\theta_0 + \theta_\infty}{2x}.$$

Combining this with (6.37) *and* (6.39)*, we arrive at the equation,*

$$Hdx = vdu - Hdx + d\left(Hx + \frac{1 - 4\theta_\infty}{4}\ln k - \frac{1 + 4\theta_0}{4}\ln a + \frac{\theta_0 + \theta_\infty}{2}\ln x\right),$$

where $df \equiv d_x f = \frac{df}{dx}$*. In other words, although the truncated action,* Hdx*, is not in this case exactly the Jimbo-Miwa-Ueno form* ω_{JMU}*, we still can conclude from the calculations above the fact that it coincides with the full classical action, up to a total differential. As we will see, this is true in all other examples when accidentally* $Hdx \neq \omega_{\mathrm{JMU}}$ *.*

We want now to check Conjecture 5.1. Substitution of $G^{(\infty,0)}(z)$ from (6.26) and (6.28) into (5.1) leads us to the formula

$$\omega_a = \mathrm{Tr}\Big(A_{0,0}\delta G_0 G_0^{-1} + G_0^{-1} A_{0,-1} G_0 \delta g_{0,1} - A_{\infty,-1}\delta g_{\infty,1}\Big).$$

Using (6.27), (6.29), (6.31) we arrive at the equation,

$$\omega_a = v\delta u - \theta_\infty \frac{\delta k}{k} - \theta_0 \frac{\delta a}{a}.$$

and taking differential δ we get

$$\delta\omega_a = \delta v \wedge \delta u + \delta(\ln k) \wedge \delta\theta_\infty + \delta(\ln a) \wedge \delta\theta_0. \tag{6.40}$$

Therefore Conjecture 5.1 holds.

For Conjecture 5.2 we can notice that in case of Painlevé-III equation second term in (5.7) starts playing role. We have

$$\Omega = \mathrm{Tr}\Big(-\frac{\partial A_{\infty,-1}}{\partial x}\delta g_{\infty,1} + \frac{\partial A_{0,0}}{\partial x}\delta G_0 G_0^{-1} + \frac{\partial A_{0,-1}}{\partial x} G_0 \delta g_{0,1} G_0^{-1} - \delta\theta_0 \sigma_3 G_0^{-1}\frac{\partial G_0}{\partial x}\Big).$$

After some computation using (6.27), (6.29) and (6.31) we conclude that (5.8) holds with H given by (6.30). That means that Conjecture 5.2 is true. We can easily check that system (6.34) is Hamiltonian system with symplectic form (6.40) and the Hamiltonian (6.30).

6.4 PIV equation

This time (see again [76]) the linear system is the 2×2 system with one irregular singular point at $z=\infty$ of Poincaré rank 2 and one Fuchsian point at $z=0$

$$\frac{d\Phi}{dz} = A(z)\,\Phi, \qquad A(z) = \frac{A_{0,0}}{z} + A_{\infty,-1} + A_{\infty,-2} z, \tag{6.41}$$

where

$$A_{\infty,-2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_{\infty,-1} = \begin{pmatrix} x & k \\ -\frac{u(4v-u-2x)+4\theta_\infty}{2k} & -x \end{pmatrix},$$
$$A_{0,0} = \frac{1}{2}\begin{pmatrix} \frac{u(4v-u-2x)}{2} & -ku \\ \frac{u^2(4v-u-2x)^2-16\theta_0^2}{4ku} & -\frac{u(4v-u-2x)}{2} \end{pmatrix}.$$

The corresponding formal solution at $z=\infty$ is

$$\Phi_{\text{form}}^{(\infty)}(z) = \left(I + \frac{g_{\infty,1}}{z} + \frac{g_{\infty,2}}{z^2} + O\left(\frac{1}{z^3}\right)\right)e^{\Theta(z)}, \quad \Theta(z) = \sigma_3\left(\frac{z^2}{2} + xz - \theta_\infty \ln z\right) \tag{6.42}$$

with

$$g_{\infty,1} = \frac{1}{2}\begin{pmatrix} -\frac{2H+u}{2} & -k \\ -\frac{u(4v-u-2x)+4\theta_\infty}{2k} & \frac{2H+u}{2} \end{pmatrix},$$
$$g_{\infty,2} = \frac{1}{8}\begin{pmatrix} \frac{\left((2H+u+2x)^2-4x^2-8\theta_0^2+8\theta_\infty^2\right)}{4} & -k\,(2H-u-4x) \\ \frac{((2H-u)(u(4v-u-2x)+4\theta_\infty+4)+8u)}{2k} & \frac{\left((2H+u+2x)^2-4x^2+8\theta_0^2-8\theta_\infty^2\right)}{4}, \end{pmatrix}, \tag{6.43}$$

and

$$H = 2v^2u - \frac{1}{8}u^3 - \frac{1}{2}xu^2 + \frac{1}{2}(2\theta_\infty - 1 - x^2)u + 2\theta_\infty x - \frac{2\theta_0^2}{u}. \tag{6.44}$$

The behavior of the solution of (6.41) at the (non-resonant, this time) Fuchsian point $z = 0$ is described by the equation,

$$\Phi^{(0)}_{\text{form}}(z) = G_0\left(I + O\left(z\right)\right) z^{\theta_0\sigma_3}, \quad z \to 0, \tag{6.45}$$

where G_0 diagonalizes the matrix $A_{0,0}$,

$$G_0^{-1}A_{0,0}G_0 = \theta_0\sigma_3$$

and it is chosen in the form,

$$G_0 = \frac{1}{2\sqrt{ku\theta_0}}\begin{pmatrix} -ku & -ku \\ -\frac{u(4v-u-2x)-4\theta_0}{2} & -\frac{u(4v-u-2x)+4\theta_0}{2} \end{pmatrix} a^{-\frac{\sigma_3}{2}}. \tag{6.46}$$

The full parameter space,

$$\mathcal{A} = \left\{v, u, k, a, x, \theta_0, \theta_\infty, \right\},$$

is again seven dimensional with x being the isomonodromic time and θ_∞ and θ_0 serving as the formal monodromy exponents at the respective singular points. The dimension is given by formula (2.2).

The isomonodromicity with respect to x yields the second differential equation for $\Phi(z)$,

$$\frac{d\Phi}{dx} = U\left(z\right)\Phi, \qquad U\left(z\right) = U_1 z + U_0, \tag{6.47}$$

where

$$U_1 = A_{\infty,-2}, \quad U_0 = \begin{pmatrix} 0 & k \\ \frac{u(4v-u-2x)+4\theta_\infty}{2k} & 0 \end{pmatrix}.$$

and the compatibility of (6.47) and (6.41) implies,

$$\begin{gathered} \frac{du}{dx} = 4vu, \quad \frac{dv}{dx} = -2v^2 + \frac{3}{8}u^2 + ux + \frac{1}{2}x^2 - \theta_\infty + \frac{1}{2} - \frac{2\theta_0^2}{u^2}, \\ \frac{dk}{dx} = -(u + 2x)k, \quad \frac{da}{dx} = \frac{4\theta_0}{u}a, \quad \frac{d\theta_\infty}{dx} = 0, \quad \frac{d\theta_0}{dx} = 0. \end{gathered} \tag{6.48}$$

As in the previous section, the fourth equation follows from the substitution of (6.45) into (6.47).

Similar to the previous cases, the last equations of (6.48) manifest the time-independence of the formal monodromy exponents, the third and the fourth equations express k and a in terms of v and u, while the first and the second equations are equivalent to a Painlevé equation, this time to the PIV equation, with

$$\alpha = 2\theta_\infty - 1, \quad \beta = -8\theta_0^2.$$

The general formulae (2.12) and (2.16) transform, in the case of system (6.41), into the equations

$$\omega_{\mathrm{JMU}} = -\operatorname{res}_{z=\infty} \operatorname{Tr}\left(\left(G^{(\infty)}(z)\right)^{-1} \frac{dG^{(\infty)}(z)}{dz} \frac{d\Theta_\infty(z)}{dx}\right) dx \tag{6.49}$$

and

$$\omega = \operatorname{res}_{z=\infty} \operatorname{Tr}\left(A(z)\, dG^{(\infty)}(z)\, G^{(\infty)}(z)^{-1}\right) + \operatorname{res}_{z=0} \operatorname{Tr}\left(A(z)\, dG^{(0)}(z)\, G^{(0)}(z)^{-1}\right), \tag{6.50}$$

respectively. Substituting the series $G^{(\infty)}(z)$ and the exponentials $\Theta_\infty(z)$ from (6.42) into (6.49), and using (6.43) we obtain that

$$\omega_{\mathrm{JMU}} \equiv \frac{d\ln\tau}{dx} dx = H dx + \frac{1}{2} u dx. \tag{6.51}$$

Note again the additional to Hdx term in the right hand side of (6.51).

Similar substitution of $G^{(\infty,0)}(z)$ from (6.42) and (6.45) into (6.50) followed by the use of (6.43) and (6.46) leads us to the formulae,

$$\begin{aligned} \omega &= \operatorname{Tr}(A_{-1} dG_0 G_0^{-1} - A_1 dg_2 + A_1 dg_1 g_1 - A_0 dg_1) \\ &= -\frac{1}{2} u dv + \frac{1}{2} v du + \frac{1}{2} x dH - \frac{1}{2} H dx - \theta_\infty \frac{dk}{k} - \theta_0 \frac{da}{a} + \theta_0 d\theta_0 - \frac{2\theta_\infty - 1}{2} d\theta_\infty. \end{aligned}$$

Regrouping the last equation, we arrive at the final answer for the form ω,

$$\omega = v du - H dx + d\left(\frac{Hx}{2} - \frac{uv}{2} - \theta_\infty \ln k - \theta_0 \ln a + \frac{\theta_0^2}{2} + \frac{\theta_\infty}{2} - \frac{\theta_\infty^2}{2}\right) + \ln k \ d\theta_\infty + \ln a \ \ d\theta_0$$

or, using the definition of the canonical coordinates,

$$p_1 = v, \quad q_1 = u, \quad p_2 = \ln k, \quad q_2 = \theta_\infty, \quad p_3 = \ln a, \quad q_3 = \theta_0,$$

we have

$$\omega = p_1 dq_1 + p_2 dq_2 + p_3 dq_3 - H dx + d\left(\frac{Hx}{2} - \frac{p_1 q_1}{2} - q_2 p_2 - q_3 p_3 + \frac{q_3^2}{2} + \frac{q_2}{2} - \frac{q_2^2}{2}\right). \tag{6.52}$$

Equation (6.52) proves Conjecture 2.2, in the case of the 2×2 system (6.41) and gives the explicit formula for $G(p_j, q_j, t)$,

$$G(p_1, p_2, p_3, q_1, q_2, q_3, x) = \frac{Hx}{2} - \frac{p_1 q_1}{2} - q_2 p_2 - q_3 p_3 + \frac{q_3^2}{2} + \frac{q_2}{2} - \frac{q_2^2}{2}.$$

The corresponding equation (2.32) and the formula for the truncated action are

$$\frac{d \ln \tau}{dx} = v \frac{du}{dx} - H + \frac{d}{dx}\left(\frac{Hx}{2} - \frac{uv}{2} - \theta_\infty \ln k - \theta_0 \ln a\right),$$

and

$$H dx = v du - H dx + d\left(\frac{Hx}{2} - \frac{uv}{2} + \frac{1 - 2\theta_\infty}{2} \ln k - \theta_0 \ln a + \frac{x^2}{2}\right), \quad d \equiv d_x,$$

respectively.

We want to check Conjecture 5.1. Substitution of $G^{(\infty,0)}(z)$ from (6.42) and (6.45) into (5.1) leads us to the formulae

$$\omega_a = \mathrm{Tr}(A_{0,0} \delta G_0 G_0^{-1} - A_{\infty,-2} d g_{\infty,2} + A_{\infty,-2} d g_{\infty,1} g_{\infty,1} - A_{\infty,-1} d g_{\infty,1}).$$

The use of (6.43) and (6.46) gives us

$$\omega_a = -\frac{1}{2} u \delta v + \frac{1}{2} v \delta u + \frac{1}{2} x \delta H - \theta_\infty \frac{\delta k}{k} - \theta_0 \frac{\delta a}{a} + \theta_0 \delta \theta_0 - \frac{2\theta_\infty - 1}{2} \delta \theta_\infty.$$

After taking differential δ we get

$$\delta \omega_a = \delta v \wedge \delta u + \delta(\ln k) \wedge \delta \theta_\infty + \delta(\ln a) \wedge \delta \theta_0 \qquad (6.53)$$

Therefore Conjecture 5.1 holds.

Now for the form (5.7) we have

$$\Omega = \mathrm{Tr}\Big(-\frac{\partial A_{\infty,-1}}{\partial x} \delta g_{\infty,1} + \frac{\partial A_{0,0}}{\partial x} \delta G_0 G_0^{-1} - \delta \theta_0 \sigma_3 G_0^{-1} \frac{\partial G_0}{\partial x}\Big).$$

After using (6.43) and (6.46) we get (5.8) with H given by (6.44). That means that Conjecture 5.2 holds. We can easily check that system (6.48) is Hamiltonian system with symplectic form (6.53) and the Hamiltonian (6.44).

6.5 PV equation

In [76] the following linear system is associated with the fifth Painlevé equation

$$\frac{d\Phi}{dz} = A(z)\,\Phi, \qquad A(z) = A_{\infty,-1} + \frac{A_{0,0}}{z} + \frac{A_{1,0}}{z-1}, \tag{6.54}$$

where

$$A_{\infty,-1} = \frac{x}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_{0,0} = \begin{pmatrix} -vu-\theta_\infty-\theta_1 & k(vu+\theta_\infty+\theta_1-\theta_0) \\ -\frac{1}{k}(vu+\theta_\infty+\theta_1+\theta_0) & vu+\theta_\infty+\theta_1 \end{pmatrix},$$

$$A_{1,0} = \begin{pmatrix} vu+\theta_1 & -ku(vu+2\theta_1) \\ \frac{v}{k} & -vu-\theta_1 \end{pmatrix}.$$

This system has one irregular singular point of Poincaré rank 1 at $z=\infty$ and two Fuchsian singular points $z=0$ and $z=1$.

The corresponding formal solution at $z=\infty$ is given by the formulae,

$$\Phi^{(\infty)}_{\text{form}}(z) = \left(I + \frac{g_1}{z} + O\left(\frac{1}{z^2}\right)\right)e^{\Theta(z)}, \quad z\to\infty, \quad \Theta(z) = \sigma_3\left(\frac{xz}{2} - \theta_\infty \ln z\right) \tag{6.55}$$

with

$$g_1 = \begin{pmatrix} -H & \dfrac{k(vu^2-vu+2\theta_1 u-\theta_\infty-\theta_1+\theta_0)}{x} \\ -\dfrac{2vu-2v+2\theta_\infty+2\theta_1+2\theta_0}{xk} & H \end{pmatrix} \tag{6.56}$$

and

$$\begin{aligned} H = {}& \frac{v^2(u-1)^2u}{x} + v\left(\frac{u^2}{x}(\theta_0+3\theta_1+\theta_\infty) + \frac{u}{x}(x-2\theta_\infty-4\theta_1) + \frac{1}{x}(\theta_\infty+\theta_1-\theta_0)\right) \\ & + \frac{u2\theta_1}{x}(\theta_\infty+\theta_1+\theta_0) + \frac{\theta_0^2-\theta_1^2-\theta_\infty^2+\theta_1 x-2\theta_1\theta_\infty}{x}. \end{aligned} \tag{6.57}$$

The behavior of the solutions of (6.54) at the (non-resonant) Fuchsian points $z=0$ and $z=1$ are described by the equations,

$$\Phi^{(0)}_{\text{form}}(z) = G_0\,(I+O(z))\,z^{\theta_0\sigma_3}, \quad z\to 0, \tag{6.58}$$

and

$$\Phi^{(1)}_{\text{form}}(z) = G_1\,(I+O(z-1))\,(z-1)^{\theta_1\sigma_3}, \quad z\to 1, \tag{6.59}$$

respectively. The matrices G_0 and G_1 diagonalize the matrix coefficients $A_{0,0}$ and $A_{1,0}$,

$$G_0^{-1}A_{0,0}G_0 = \theta_0\sigma_3, \quad G_1^{-1}A_{1,0}G_1 = \theta_1\sigma_3,$$

and are chosen in the form,

$$G_0 = \frac{1}{\sqrt{-4k\theta_0}}\begin{pmatrix} k(2vu+2\theta_\infty+2\theta_1-2\theta_0) & k \\ 2vu+2\theta_\infty+2\theta_1+2\theta_0 & 1 \end{pmatrix} a^{-\frac{\sigma_3}{2}}, \tag{6.60}$$

and

$$G_1 = \frac{1}{\sqrt{2k\theta_1}}\begin{pmatrix} k(vu+2\theta_1) & ku \\ v & 1 \end{pmatrix} b^{-\frac{\sigma_3}{2}}. \tag{6.61}$$

The full space

$$\mathcal{A} = \{v, u, k, a, b, x, \theta_0, \theta_1, \theta_\infty, \}\,.$$

is nine dimensional with x being the isomonodromic time and θ_∞, θ_0, and θ_1 serving as the formal monodromy exponents at the respective singular points. The dimension is given by (2.2).

The isomonodromicity with respect to x yields the second differential equation for $\Phi(z)$,

$$\frac{d\Phi}{dx} = U\,(z)\,\Phi, \qquad U\,(z) = U_1 z + U_0, \tag{6.62}$$

where

$$U_1 = \frac{A_{\infty,-1}}{x},\; U_0 = \begin{pmatrix} 0 & \frac{k}{x}(-vu^2+vu-2\theta_1 u+\theta_\infty+\theta_1-\theta_0) \\ -\frac{1}{xk}(vu+\theta_\infty-v+\theta_1+\theta_0) & 0 \end{pmatrix},$$

and the compatibility of (6.62) and (6.54) implies,

$$\begin{aligned}
\frac{du}{dx} &= \frac{2vu(u-1)^2}{x} + \frac{u^2}{x}(\theta_0+3\theta_1+\theta_\infty) + \frac{u}{x}(x-2\theta_\infty-4\theta_1) + \frac{1}{x}(\theta_\infty+\theta_1-\theta_0),\\
\frac{dv}{dx} &= -\frac{v^2}{x}(3u^2-4u+1) - v\left(\frac{2u}{x}(\theta_0+3\theta_1+\theta_\infty) + \frac{1}{x}(x-2\theta_\infty-4\theta_1)\right) - \frac{2\theta_1}{x}(\theta_\infty+\theta_1+\theta_0),\\
\frac{dk}{dx} &= -\frac{k}{x}(vu^2-2vu+v+2\theta_1 u-2\theta_\infty-2\theta_1),\\
\frac{da}{dx} &= \frac{a}{x}(v-vu^2-2\theta_1 u-2\theta_0),\\
\frac{db}{dx} &= -\frac{b}{x}(3vu^2+v-4vu+2\theta_\infty u+4\theta_1 u+2\theta_0 u-2\theta_\infty-2\theta_1+x),\\
&\frac{d\theta_\infty}{dx} = 0, \quad \frac{d\theta_0}{dx} = 0, \quad \frac{d\theta_1}{dx} = 0.
\end{aligned} \tag{6.63}$$

As before, the equations for a and b follow from the substitution of (6.58) into (6.62) and (6.59) into (6.62), respectively, and they simply express the functions $a(x)$ and $b(x)$ in terms of v and u. The third equation in (6.63) is also trivial – just an expression of k in terms of v and u, and the last three equations are the manifestation of the time-independence of the formal monodromy exponents. The nontrivial first two equations are equivalent to the PV equation, with

$$\alpha = \frac{(\theta_0 - \theta_1 + \theta_\infty)^2}{2}, \quad \beta = -\frac{(\theta_0 - \theta_1 - \theta_\infty)^2}{2}, \quad \gamma = (1 - 2\theta_0 - 2\theta_1), \quad \delta = -\frac{1}{2}.$$

The general formulae (2.12) and (2.16) transform, in the case of system (6.54), into the equations

$$\omega_{\text{JMU}} = -\operatorname{res}_{z=\infty} \operatorname{Tr}\left(\left(G^{(\infty)}(z)\right)^{-1} \frac{dG^{(\infty)}(z)}{dz} \frac{d\Theta_\infty(z)}{dx}\right) dx \tag{6.64}$$

and

$$\begin{aligned} \omega = \operatorname{res}_{z=\infty} \operatorname{Tr}\left(A(z)\, dG^{(\infty)}(z)\, G^{(\infty)}(z)^{-1}\right) + \operatorname{res}_{z=0} \operatorname{Tr}\left(A(z)\, dG^{(0)}(z)\, G^{(0)}(z)^{-1}\right) \\ + \operatorname{res}_{z=1} \operatorname{Tr}\left(A(z)\, dG^{(1)}(z)\, G^{(1)}(z)^{-1}\right), \end{aligned} \tag{6.65}$$

respectively. Substituting the series $G^{(\infty)}(z)$ and the exponent $\Theta_\infty(z)$ from (6.55) into (6.64) and using (6.56), we obtain that

$$\omega_{\text{JMU}} \equiv \frac{d \ln \tau}{dx} dx = H dx. \tag{6.66}$$

Substituting $G^{(\infty,0,1)}(z)$ from (6.55), (6.58), and (6.59) into (6.65) and using after that (6.56), (6.60), (6.61) leads us to the formulae,

$$\begin{aligned} \omega &= \operatorname{Tr}(A_0 dG_0 G_0^{-1} + A_1 dG_1 G_1^{-1} - A_2 dg_1) \\ &= pdq + xdH - \theta_\infty \frac{dk}{k} - \theta_0 \frac{da}{a} - \theta_1 \frac{db}{b} + d\theta_0 + d\theta_1. \end{aligned}$$

Regrouping the last equation, we arrive at the final answer for the form ω,

$$\omega = vdu - Hdx + d\Big(Hx - \theta_\infty \ln k - \theta_0 \ln a - \theta_1 \ln b + \theta_0 + \theta_1\Big) + \ln a\ d\theta_0 + \ln b\ d\theta_1 + \ln k\ d\theta_\infty,$$

or, using the definition of the canonical coordinates,

$$p_1 = v, \quad q_1 = u, \quad p_2 = \ln k, \quad q_2 = \theta_\infty, \quad p_3 = \ln a, \quad q_3 = \theta_0, \quad p_4 = \ln b, \quad q_4 = \theta_1,$$

we have

$$\omega = p_1dq_1 + p_2dq_2 + p_3dq_3 + p_4dq_4 - Hdx + d\Big(Hx - q_2p_2 - q_3p_3 - q_4p_4 + q_3 + q_4\Big). \quad (6.67)$$

Equation (6.67) proves Conjecture 2.2, in the case of the 2×2 system (6.54) and gives the explicit formula for $G(p_j, q_j, x)$,

$$G(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4, x) = Hx - q_2p_2 - q_3p_3 - q_4p_4 + q_3 + q_4.$$

The corresponding equation (2.32) is

$$\frac{d\ln\tau}{dx} = v\frac{du}{dx} - H + \frac{d}{dx}\Big(Hx - \theta_\infty \ln k - \theta_0 \ln a - \theta_1 \ln b\Big).$$

This equation together with (6.66) make an identity, which would not be so easy to check directly.

We want to check Conjecture 5.1 now. Substituting $G^{(\infty,0,1)}(z)$ from (6.55), (6.58), and (6.59) into (5.1) leads us to the formulae,

$$\omega_a = \mathrm{Tr}(A_{0,0}\delta G_0 G_0^{-1} + A_{1,0}\delta G_1 G_1^{-1} - A_{\infty,-1}\delta g_1).$$

Using after that (6.56), (6.60), (6.61) we get

$$\omega_a = v\delta u + x\delta H - \theta_\infty\frac{\delta k}{k} - \theta_0\frac{\delta a}{a} - \theta_1\frac{\delta b}{b} + \delta\theta_0 + \delta\theta_1.$$

After taking differential δ we obtain

$$\delta\omega_a = \delta v \wedge \delta u + \delta(\ln k) \wedge \delta\theta_\infty + \delta(\ln a) \wedge \delta\theta_0 + \delta(\ln b) \wedge \delta\theta_1 \quad (6.68)$$

Therefore Conjecture 5.1 holds.

Now for the form (5.7) we have

$$\Omega = \mathrm{Tr}\Big(-\frac{\partial A_{-1}}{\partial x}\delta g_1\Big) = \delta H$$

with H given by (6.57). That means that Conjecture 5.2 holds. We can easily check that system (6.63) is Hamiltonian system with symplectic form (6.68) and the Hamiltonian (6.57).

6.6 PVI equation

Consider the 2×2 Fuchsian system with 4 regular singularities at $0, 1, x$ and ∞

$$\frac{d\Phi}{dz} = A(z)\,\Phi, \qquad A(z) = \frac{A_0}{z} + \frac{A_x}{z-x} + \frac{A_1}{z-1}, \tag{6.69}$$

where

$$A_0, A_1, A_x \in \mathfrak{sl}_2(\mathbb{C}), \quad A_0 + A_1 + A_x = -\theta_\infty \sigma_3. \tag{6.70}$$

Following to [76], we introduce the parametrization

$$A_0 = \begin{pmatrix} y_0 + \theta_0 & -ly_0 \\ \frac{y_0+2\theta_0}{l} & -y_0 - \theta_0 \end{pmatrix}, \; A_1 = \begin{pmatrix} y_1 + \theta_1 & -my_1 \\ \frac{y_1+2\theta_1}{m} & -y_1 - \theta_1 \end{pmatrix}, \; A_x = \begin{pmatrix} y_x + \theta_x & -ny_x \\ \frac{y_x+2\theta_x}{n} & -y_x - \theta_x \end{pmatrix}. \tag{6.71}$$

Observe that, $\pm\theta_0, \pm\theta_1, \pm\theta_x$ are the eigenvalues of A_0, A_1, A_x, and that the following constraints are satisfied because of (6.70)

$$y_0 + \theta_0 + y_1 + \theta_1 + y_x + \theta_x = -\theta_\infty, \tag{6.72}$$

$$ly_0 + my_1 + ny_x = 0, \tag{6.73}$$

$$\frac{y_0 + 2\theta_0}{l} + \frac{y_1 + 2\theta_1}{m} + \frac{y_x + 2\theta_x}{n} = 0. \tag{6.74}$$

We also introduce the parameters k and u by writing the entry $A_{12}(z)$ of the matrix $A(z)$ as,

$$A_{12}(z) = \frac{k(z-u)}{z(z-1)(z-x)}.$$

Notice that,

$$ly_0(1+x) + my_1 + ny_x = k, \quad ly_0 x = ku. \tag{6.75}$$

Finally we put

$$v = A_{11}(u) = \frac{y_0 + \theta_0}{u} + \frac{y_1 + \theta_1}{u-1} + \frac{y_x + \theta_x}{u-x}. \tag{6.76}$$

Solving equations (6.73) and (6.75) with respect to u, v, w, we get

$$l = \frac{ku}{y_0 x}, \quad m = \frac{k(u-1)}{y_1(1-x)}, \quad n = \frac{k(x-u)}{y_x x(1-x)}. \tag{6.77}$$

Next, we express y_1, y_x from (6.72), (6.76), and then we express y_0 from (6.74). The result is

$$
\begin{aligned}
y_0 = {} & \frac{v^2u^2(u-1)(u-x)}{x2\theta_\infty} + \frac{vu(u-1)(u-x)}{x} + \frac{\theta_\infty u(u-x-1)}{2x} + \frac{\theta_1^2(x-1)}{2x\theta_\infty(u-1)} \\
& - \frac{\theta_x^2 x(x-1)}{2\theta_\infty(u-x)} - \frac{\theta_1^2(1-x)}{2x\theta_\infty} - \frac{\theta_x^2 x(x-1)}{2x\theta_\infty} - \theta_0 - \frac{\theta_0^2}{2\theta_\infty}, \\
y_1 = {} & \frac{v^2u(u-1)^2(x-u)}{(x-1)2\theta_\infty} + \frac{vu(u-1)(x-u)}{x-1} + \frac{\theta_\infty(u-1)(x-u-1)}{2(x-1)} - \frac{\theta_0^2 x}{2u\theta_\infty(x-1)} \\
& + \frac{\theta_x^2 x(x-1)}{2\theta_\infty(u-x)} + \frac{\theta_0^2 x}{2(x-1)\theta_\infty} + \frac{\theta_x^2 x(x-1)}{2(x-1)\theta_\infty} - \theta_1 - \frac{\theta_1^2}{2\theta_\infty}, \\
y_x = {} & \frac{v^2u(u-1)(x-u)^2}{x(x-1)2\theta_\infty} + \frac{vu(u-1)(u-x)}{x(x-1)} + \frac{\theta_\infty(u-x)(u+x-1)}{2x(x-1)} + \frac{\theta_0^2 x}{2u\theta_\infty(x-1)} \\
& - \frac{\theta_1^2(x-1)}{2\theta_\infty x(u-1)} - \frac{\theta_0^2}{2(x-1)\theta_\infty} + \frac{\theta_1^2}{2x\theta_\infty} - \theta_x - \frac{\theta_x^2}{2\theta_\infty}.
\end{aligned}
\tag{6.78}
$$

Equations (6.77) – (6.78) provide parametrization of the matrices A_0, A_1, and A_x by the variables $u, v, k, \theta_0, \theta_1, \theta_x, \theta_\infty$, which will prove to be the Darboux coordinates, and by the parameter x which is the isomonodromic time.

Solutions of (6.69) have the following behavior at $z = 0, 1, x$, and ∞

$$
\begin{aligned}
\Phi^{(\infty)}_{\text{form}}(z) &= \left(I + O\left(z^{-1}\right)\right) z^{-\theta_\infty\sigma_3}, \quad z \to \infty. \\
\Phi^{(0)}_{\text{form}}(z) &= G_0\left(I + O\left(z\right)\right) z^{\theta_0\sigma_3}, \quad z \to 0, \\
\Phi^{(1)}_{\text{form}}(z) &= G_1\left(I + O\left(z-1\right)\right) (z-1)^{\theta_1\sigma_3}, \quad z \to 1, \\
\Phi^{(x)}_{\text{form}}(z) &= G_x\left(I + g_1(z-x) + O\left((z-x)^2\right)\right) (z-x)^{\theta_x\sigma_3}, \quad z \to x.
\end{aligned}
\tag{6.79}
$$

The matrices G_0, G_1, and G_x diagonalize the matrix residues A_0, A_1, and A_x,

$$
G_0^{-1}A_0G_0 = \theta_0\sigma_3, \quad G_1^{-1}A_1G_1 = \theta_1\sigma_3, \quad G_x^{-1}A_xG_x = \theta_x\sigma_3.
$$

and they are chosen in the form,

$$
\begin{aligned}
G_0 &= \sqrt{\frac{ku}{2\theta_0 x}} \begin{pmatrix} 1 & 1 \\ \frac{1}{l} & \frac{y_0+2\theta_0}{ly_0} \end{pmatrix} a^{-\frac{\sigma_3}{2}}, \\
G_1 &= \sqrt{\frac{k(u-1)}{2\theta_1(1-x)}} \begin{pmatrix} 1 & 1 \\ \frac{1}{m} & \frac{y_1+2\theta_1}{my_1} \end{pmatrix} b^{-\frac{\sigma_3}{2}}, \\
G_x &= \sqrt{\frac{k(x-u)}{2\theta_x x(1-x)}} \begin{pmatrix} 1 & 1 \\ \frac{1}{n} & \frac{y_x+2\theta_x}{ny_x} \end{pmatrix} c^{-\frac{\sigma_3}{2}}.
\end{aligned}
\tag{6.80}
$$

We also have

$$
\begin{aligned}
(g_1)_{11} &= \frac{H}{2\theta_x} - v\frac{u(u-1)}{2\theta_x x(x-1)} - \frac{\theta_\infty(u-x)}{2\theta_x x(x-1)} \\
(g_1)_{12} &= \frac{Hc}{2\theta_x(1-2\theta_x)} - v\frac{u(u-1)c}{2\theta_x x(x-1)} - \frac{\theta_\infty(u-x)c}{2\theta_x x(x-1)} - \frac{\theta_x(2ux-x^2-u)c}{(2\theta_x-1)x(x-1)(u-x)} \\
(g_1)_{21} &= \frac{H}{2\theta_x(1+2\theta_x)c} + v\frac{u(u-1)}{2\theta_x x(x-1)c} + \frac{\theta_\infty(u-x)}{2\theta_x x(x-1)c} - \frac{\theta_x(2ux-x^2-u)}{(2\theta_x+1)x(x-1)(u-x)c} \\
(g_1)_{22} &= -\frac{H}{2\theta_x} + v\frac{u(u-1)}{2\theta_x x(x-1)} + \frac{\theta_\infty(u-x)}{2\theta_x x(x-1)}
\end{aligned}
\tag{6.81}
$$

where

$$
\begin{aligned}
H &= v^2\frac{u(u-1)(u-x)}{x(x-1)} + v\frac{u(u-1)}{x(x-1)} + \frac{\theta_\infty(1-\theta_\infty)(u-x)}{x(x-1)} \\
&+ \frac{\theta_0^2(u-x)}{ux(x-1)} - \frac{\theta_1^2(u-x)}{(u-1)x(x-1)} + \frac{\theta_x^2(x^2-u(2x-1))}{(u-x)x(x-1)}.
\end{aligned}
\tag{6.82}
$$

The whole parameter space $\mathcal{A}$ has dimension 11 as given by (2.2),

$$
\mathcal{A} = \{v, u, k, a, b, c, x, \theta_0, \theta_1, \theta_t, \theta_\infty\}. \tag{6.83}
$$

The isomonodromicity with respect to x yields the second differential equation for $\Phi(z)$,

$$
\frac{d\Phi}{dx} = -\frac{A_x}{z-x}\Phi. \tag{6.84}
$$

and the equations

$$
\frac{dG_0}{dx} = \frac{A_x}{x}G_0, \quad \frac{dG_1}{dx} = \frac{A_x}{x-1}G_1, \quad \frac{dG_x}{dx} = \left(\frac{A_0}{x} + \frac{A_1}{x-1}\right)G_x, \tag{6.85}
$$

for the gauge matrices G_0, G_1, G_x. The compatibility of (6.84) and (6.69) together with the equations (6.85) imply the following dynamical system on (6.83),

$$
\begin{aligned}
\frac{du}{dx} &= \frac{2vu(u-1)(u-x)}{x(x-1)} + \frac{u(u-1)}{x(x-1)}, \\
\frac{dv}{dx} &= \frac{1}{4x(x-1)}\left(4v^2(2xu-3u^2-x+2u) + 4v(1-2u) + 4\theta_\infty(\theta_\infty-1)\right) \\
&\quad - \frac{\theta_0^2}{u^2(x-1)} + \frac{\theta_1^2}{x(u-1)^2} - \frac{\theta_x^2}{(u-x)^2}, \\
\frac{dk}{dx} &= \frac{k(2\theta_\infty-1)(u-x)}{x(x-1)}, \\
\frac{da}{dx} &= -\frac{2\theta_0(u-x)a}{ux(x-1)}, \quad \frac{db}{dx} = \frac{2\theta_1(u-x)b}{x(x-1)(u-1)}, \quad \frac{dc}{dx} = \frac{2\theta_x(u(2x-1)-x^2)c}{(u-x)x(x-1)}, \\
\frac{d\theta_0}{dx} &= \frac{d\theta_1}{dx} = \frac{d\theta_x}{dx} = \frac{d\theta_\infty}{dx} = 0.
\end{aligned}
\tag{6.86}
$$

As before, the only non-trivial equations are the first two, and they are equivalent to the PVI equation for the function $u(x)$, with

$$\alpha = \frac{(2\theta_\infty - 1)^2}{2}, \quad \beta = -2\theta_0^2, \quad \gamma = 2\theta_1^2, \quad \delta = \frac{1-4\theta_x^2}{2}.$$

The general formulae (2.12) and (2.16) transform, in the case of system (6.69), into the equations

$$\omega_{\mathrm{JMU}} = -\operatorname{res}_{z=t} \operatorname{Tr}\left(\left(G^{(x)}(z)\right)^{-1} \frac{dG^{(x)}(z)}{dz}\frac{d\Theta_t(z)}{dx}\right) dx \tag{6.87}$$

and

$$\begin{aligned}\omega = \operatorname{res}_{z=\infty} \operatorname{Tr}\left(A(z)\, dG^{(\infty)}(z)\, G^{(\infty)}(z)^{-1}\right) + \operatorname{res}_{z=0} \operatorname{Tr}\left(A(z)\, dG^{(0)}(z)\, G^{(0)}(z)^{-1}\right)\\ \operatorname{res}_{z=1} \operatorname{Tr}\left(A(z)\, dG^{(1)}(z)\, G^{(1)}(z)^{-1}\right) + \operatorname{res}_{z=t} \operatorname{Tr}\left(A(z)\, dG^{(x)}(z)\, G^{(x)}(z)^{-1}\right).\end{aligned} \tag{6.88}$$

From (6.87) it follows that

$$\omega_{\mathrm{JMU}} = \theta_x \operatorname{Tr}\Big(g_1\sigma_3\Big),$$

and taking into account (6.81), we obtain that,

$$\omega_{\mathrm{JMU}} \equiv \frac{d\ln\tau}{dx}dx = Hdx - v\frac{u(u-1)}{x(x-1)}dx - \frac{\theta_\infty(u-x)}{x(x-1)}dt.$$

Similarly, (6.88) reduces to the equation,

$$\omega = \operatorname{Tr}\left(\mathrm{G_0^{-1}A_0dG_0 + G_1^{-1}A_1dG_1 + G_x^{-1}A_xdG_x - A_xG_xg_1G_x^{-1}dx}\right),$$

which after using (6.80) and simplifying yields the formula

$$\omega = vdu - Hdx - \theta_\infty\frac{dk}{k} - \theta_0\frac{da}{a} - \theta_1\frac{db}{b} - \theta_x\frac{dc}{c} + d\theta_\infty.$$

This, in turn, can be rewritten as

$$\omega = vdu - Hdx + d\Big(\theta_\infty - \theta_0\ln a - \theta_1\ln b - \theta_x\ln c - \theta_\infty\ln k\Big) + \ln k d\theta_\infty + \ln a d\theta_0 + \ln b\, d\theta_1 + \ln c\, d\theta_x.$$

or, introducing the canonical coordinates,

$$p_1 = v, \quad q_1 = u, \quad p_2 = \ln k, \quad q_2 = \theta_\infty, \quad p_3 = \ln a, \quad q_3 = \theta_0,$$
$$p_4 = \ln b, \quad q_4 = \theta_1, \quad p_5 = \ln c, \quad q_5 = \theta_x.$$

$$\omega = \sum_{j=1}^{5} p_j dq_j - Hdx + d\Big(q_2 - q_3p_3 - q_4p_4 - q_5p_5 - q_2p_2\Big). \tag{6.89}$$

Equation (6.89) proves Conjecture 2.2, in the case of the 2×2 system (6.69) and gives the explicit formula for $G(p_j, q_j, x)$

$$G(p_1, p_2, p_3, p_4, p_5, q_1, q_2, q_3, q_4, q_5, x) = q_2 - q_3p_3 - q_4p_4 - q_5p_5 - q_2p_2.$$

The corresponding equation (2.32) and the truncated action are

$$\frac{d\ln\tau}{dx} = v\frac{du}{dx} - H - \frac{d}{dx}\Big(\theta_0 \ln a + \theta_1 \ln b + \theta_x \ln c + \theta_\infty \ln k\Big),$$

and

$$Hdx = vdu - Hdx + d\Big(\frac{1}{2}\ln\left(\frac{k(q-x)}{x(x-1)}\right) - \theta_0 \ln a - \theta_1 \ln b - \theta_x \ln c - \theta_\infty \ln k\Big), \quad d \equiv d_x,$$

respectively.

We now want to check Conjecture 5.1. Substituting $G^{(\infty,0,1,x)}(z)$ from (6.79) into (5.1) leads us to the formula

$$\omega_a = \mathrm{Tr}\left(\mathrm{G}_0^{-1}\mathrm{A}_0\delta\mathrm{G}_0 + \mathrm{G}_1^{-1}\mathrm{A}_1\delta\mathrm{G}_1 + \mathrm{G}_\mathrm{x}^{-1}\mathrm{A}_\mathrm{x}\delta\mathrm{G}_\mathrm{x}\right),$$

which after using (6.71), (6.80) and simplifying yields the formula

$$\omega_a = v\delta u - \theta_\infty\frac{\delta k}{k} - \theta_0\frac{\delta a}{a} - \theta_1\frac{\delta b}{b} - \theta_x\frac{\delta c}{c} + \delta\theta_\infty.$$

After taking differential δ we get

$$\delta\omega_a = \delta v \wedge \delta u + \delta(\ln k) \wedge \delta\theta_\infty + \delta(\ln a) \wedge \delta\theta_0 + \delta(\ln b) \wedge \delta\theta_1 + \delta(\ln c) \wedge \delta\theta_x. \tag{6.90}$$

Therefore Conjecture 5.1 holds.

Now for the form (5.7) we have

$$\Omega = \mathrm{Tr}\Big(\frac{\partial A_0}{\partial x}\delta G_0 G_0^{-1} + \frac{\partial A_1}{\partial x}\delta G_1 G_1^{-1} + \frac{\partial A_x}{\partial x}\delta G_x G_x^{-1} + A_x G_x \delta g_1 G_x^{-1}$$
$$-\delta\theta_0\sigma_3 G_0^{-1}\frac{\partial G_0}{\partial x} - \delta\theta_1\sigma_3 G_1^{-1}\frac{\partial G_1}{\partial x} - \delta\theta_x\sigma_3 G_x^{-1}\frac{\partial G_x}{\partial x} + \delta\theta_x\sigma_3 g_1\Big).$$

After some computation we get (5.8) with H given by (6.82). That means that Conjecture 5.2 holds. We can easily check that system (6.86) is Hamiltonian system with symplectic form (6.90) and the Hamiltonian (6.82).

REFERENCES

[1] M. J. Ablowitz and P. A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge University Press, 1991.

[2] M. V. Babich. On canonical parametrization of the phase spaces of equations of isomonodromic deformations of Fuchsian systems of dimension 2×2. Derivation of the Painlevé VI equation . *Russian Mathematical Surveys*, 64(1):45–127, 2009.

[3] M. V. Babich and L. A. Bordag. Projective Differential Geometrical Structure of the Painlevé Equations. *Journal of Differential Equations*, 157(2):452 – 485, 1999.

[4] J. Baik, R. Buckingham, and J. DiFranco. Asymptotics of Tracy-Widom Distributions and the Total Integral of a Painlevé II Function. *Communications in Mathematical Physics*, 280(2):463–497, 2008.

[5] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *Journal of the American Mathematical Society*, 12(4):1119–1178, 1999.

[6] E. Barouch, B. M. McCoy, C. A. Tracy, and T. T. Wu. Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region. *Phys. Rev. B*, 13:316–374, 1976.

[7] E. Barouch, B. M. McCoy, and T. T. Wu. Zero-Field Susceptibility of the Two-Dimensional Ising Model near T_c. *Phys. Rev. Lett.*, 32:35–35, 1974.

[8] E. L. Basor and C. A. Tracy. Asymptotics of a tau-function and Toeplitz determinants with singular generating functions. *International Journal of Modern Physics A*, 07:83–107, 1992.

[9] M. A. Bershtein and A. I. Shchechkin. Bilinear Equations on Painlevé τ Functions from CFT. *Communications in Mathematical Physics*, 339(3):1021–1061, 2015.

[10] M. Bertola. CORRIGENDUM: The dependence on the monodromy data of the isomonodromic tau function. arXiv:1601.04790 [math-ph].

[11] M. Bertola. The Dependence on the Monodromy Data of the Isomonodromic Tau Function. *Communications in Mathematical Physics*, 294(2):539–579, 2010.

[12] M. Bertola and D. Korotkin. Symplectic extensions of the Kirillov-Kostant and Goldman Poisson structures and Fuchsian systems. arXiv:1903.09197 [math-ph].

[13] A. Bloemendal and B. Virág. Limits of spiked random matrices I. *Probability Theory and Related Fields*, 156(3):795–825, 2013.

[14] P. Boalch. Symplectic Manifolds and Isomonodromic Deformations. *Advances in Mathematics*, 163(2):137 – 205, 2001.

[15] P. Boalch. Simply-laced isomonodromy systems. *Publications mathématiques de l'IHÉS*, 116(1):1–68, 2012.

[16] A. I. Bobenko and U. Eitner. *Painlevé Equations in the Differential Geometry of Surfaces*. Springer-Verlag Berlin Heidelberg, 2000.

[17] A. Bogatskiy, T. Claeys, and A. Its. Hankel Determinant and Orthogonal Polynomials for a Gaussian Weight with a Discontinuity at the Edge. *Communications in Mathematical Physics*, 347(1):127–162, Oct 2016.

[18] G. Bonelli, A. Grassi, and A. Tanzini. Seiberg–Witten theory as a Fermi gas. *Letters in Mathematical Physics*, 107(1):1–30, 2017.

[19] G. Bonelli, O. Lisovyy, K. Maruyoshi, A. Sciarappa, and A. Tanzini. On Painlevé/gauge theory correspondence. *Letters in Mathematical Physics*, 107(12):2359–2413, 2017.

[20] T. Bothner. A Short Note on the Scaling Function Constant Problem in the Two-Dimensional Ising Model. *Journal of Statistical Physics*, 170(4):672–683, 2018.

[21] T. Bothner and R. Buckingham. Large Deformations of the Tracy–Widom Distribution I: Non-oscillatory Asymptotics. *Communications in Mathematical Physics*, 359(1):223–263, Apr 2018.

[22] T. Bothner, A. Its, and A. Prokhorov. On the analysis of incomplete spectra in random matrix theory through an extension of the Jimbo–Miwa–Ueno differential. *Advances in Mathematics*, 345:483 – 551, 2019.

[23] T. Bothner and W. Warner. Short Distance Asymptotics for a Generalized Two-point Scaling Function in the Two-dimensional Ising Model. *Mathematical Physics, Analysis and Geometry*, 21(4):37, 2018.

[24] E. Brézin and V. Kazakov. Exactly solvable field theories of closed strings. *Physics Letters B*, 236(2):144 – 150, 1990.

[25] R. Buckingham and K. Liechty. The k-tacnode process. *Probability Theory and Related Fields*, 2018.

[26] H. Chiba. The first, second and fourth Painlevé equations on weighted projective spaces. *Journal of Differential Equations*, 260(2):1263 – 1313, 2016.

[27] H. Chiba. The Third, Fifth and Sixth Painlevé Equations on Weighted Projective Spaces. *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)*, 12:019, 2016.

[28] T. Claeys, A. Its, and I. Krasovsky. Emergence of a singularity for Toeplitz determinants and Painlevé V. *Duke Math. J.*, 160(2):207–262, 2011.

[29] R. Conte and I. Dornic. The master Painlevé VI heat equation. *Comptes Rendus Mathematique*, 352(10):803 – 806, 2014.

[30] R. Conte and M. Musette. *The Painlevé Handbook*. Springer Netherlands, 2008.

[31] B. C. da Cunha, M. C. de Almeida, and A. R. de Queiroz. On the existence of monodromies for the Rabi model. *Journal of Physics A: Mathematical and Theoretical*, 49(19):194002, 2016.

[32] D. Dai and S.-X. Xu. Tracy–Widom Distributions in Critical Unitary Random Matrix Ensembles and the Coupled Painlevé II System. *Communications in Mathematical Physics*, 365(2):515–567, 2019.

[33] P. Deift, A. Its, and I. Krasovsky. Asymptotics of the Airy-Kernel Determinant. *Communications in Mathematical Physics*, 278(3):643–678, 2008.

[34] P. Deift, A. Its, and I. Krasovsky. Toeplitz Matrices and Toeplitz Determinants under the Impetus of the Ising Model: Some History and Some Recent Results. *Communications on Pure and Applied Mathematics*, 66(9):1360–1438, 2013.

[35] P. Deift, A. Its, I. Krasovsky, and X. Zhou. The Widom–Dyson constant for the gap probability in random matrix theory. *Journal of Computational and Applied Mathematics*, 202(1):26 – 47, 2007. Special Issue: Topics on Integrable Systems.

[36] P. Deift, I. Krasovsky, and J. Vasilevska. Asymptotics for a Determinant with a Confluent Hypergeometric Kernel. *International Mathematics Research Notices*, 2011(9):2117–2160, 2010.

[37] P. Deift and X. Zhou. A Steepest Descent Method for Oscillatory Riemann–Hilbert Problems. Asymptotics for the MKdV Equation. *Annals of Mathematics*, 137(2):295–368, 1993.

[38] P. Deift and X. Zhou. Asymptotics for the Painlevé-II equation. *Communications on Pure and Applied Mathematics*, 48(3):277–337, 1995.

[39] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/.

[40] I. Dornic. Universal Painlevé VI Probability Distribution in Pfaffian Persistence and Gaussian First-Passage Problems with a sech-Kernel. arXiv:1810.06957 [cond-mat.stat-mech].

[41] M. R. Douglas and S. H. Shenker. Strings in less than one dimension. *Nuclear Physics B*, 335(3):635 – 654, 1990.

[42] B. Dubrovin. Geometry of 2D topological field theories. In M. Francaviglia and S. Greco, editors, *Integrable Systems and Quantum Groups: Lectures given at the 1st Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Montecatini Terme, Italy, June 14–22, 1993*, pages 120–348. Springer Berlin Heidelberg, Berlin, Heidelberg, 1996.

[43] B. Dubrovin, D. Yang, and D. Zagier. Classical Hurwitz Numbers and Related Combinatoricshowpublished. *Moscow Mathematical Journal*, 17(4):601–603, 2017.

[44] T. Ehrhardt. Dyson's Constant in the Asymptotics of the Fredholm Determinant of the Sine Kernel. *Communications in Mathematical Physics*, 262(2):317–341, 2006.

[45] T. Ehrhardt. The asymptotics of a Bessel-kernel determinant which arises in Random Matrix Theory. *Advances in Mathematics*, 225(6):3088 – 3133, 2010.

[46] A. Eremenko and A. Gabrielov. Circular Pentagons and Real Solutions of Painlevé VI Equations. *Communications in Mathematical Physics*, 355(1):51–95, 2017.

[47] G. V. Filipuk and V. I. Gromak. On Algebraic Solutions of the Fifth Painlevé Equation. *Differential Equations*, 39(3):322–330, 2003.

[48] H. Flaschka and A. C. Newell. Monodromy- and spectrum-preserving deformations. I. *Comm. Math. Phys.*, 76(1):65–116, 1980.

[49] A. Fokas, A. Its, A. Kapaev, and V. Novokshenov. *Painlevé Transcendents: The Riemann-Hilbert Approach.* American Mathematical Society, 2006.

[50] A. Fokas and S. Tanveer. A Hele-Shaw problem and the second Painlevé transcendent. *Mathematical Proceedings of the Cambridge Philosophical Society*, 124(1):169–191, 1998.

[51] P. J. Forrester. *Log-Gases and Random Matrices.* London Mathematical Society Monographs. Princeton University Press, 2010.

[52] P. J. Forrester and N. S. Witte. Application of the τ-Function Theory of Painlevé Equations to Random Matrices: PIV, PII and the GUE. *Communications in Mathematical Physics*, 219(2):357–398, 2001.

[53] P. J. Forrester and N. S. Witte. Application of the τ-function theory of Painlevé equations to random matrices: PV, PIII, the LUE, JUE and CUE. *Communications on Pure and Applied Mathematics*, 55(6):679–727, 2002.

[54] P. J. Forrester and N. S. Witte. Application of the τ-function theory of Painlevé equations to random matrices: P_{VI}, the JUE, CyUE, cJUE and scaled limits. *Nagoya Math. J.*, 174:29–114, 2004.

[55] O. Gamayun, N. Iorgov, and O. Lisovyy. Conformal field theory of Painlevé VI. *Journal of High Energy Physics*, 2012(10):38, 2012.

[56] O. Gamayun, N. Iorgov, and O. Lisovyy. How instanton combinatorics solves Painlevé VI, V and IIIs. *Journal of Physics A: Mathematical and Theoretical*, 46(33):335203, 2013.

[57] O. Gamayun and O. Lisovyy. On self-similar solutions of the vortex filament equation. arXiv:1903.02105 [math-ph].

[58] B. Gambier. Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est a points critiques fixes. *Acta Mathematica*, 33:1–55, 1910.

[59] V. I. Gromak, I. Laine, and S. Shimomura. *Painlevé Differential Equations in the Complex Plane.* De Gruyter, 2002.

[60] D. J. Gross and A. A. Migdal. A nonperturbative treatment of two-dimensional quantum gravity. *Nuclear Physics B*, 340(2):333 – 365, 1990.

[61] D. Guzzetti. Inverse Problem and Monodromy Data for Three-Dimensional Frobenius Manifolds. *Mathematical Physics, Analysis and Geometry*, 4(3):245–291, 2001.

[62] D. Guzzetti. Tabulation of Painlevé 6 transcendents. *Nonlinearity*, 25(12):3235–3276, 2012.

[63] J. Harnad and M. Routhier. R-matrix construction of electromagnetic models for the Painlevé transcendents. *Journal of Mathematical Physics*, 36(9):4863–4881, 1995.

[64] N. Hitchin. Hypercomplex Manifolds and the Space of Framings. In *The geometric universe : science, geometry, and the work of Roger Penrose*, pages 9–30. Oxford University Press, 1998.

[65] M. Huang, S.-X. Xu, and L. Zhang. Location of Poles for the Hastings–McLeod Solution to the Second Painlevé Equation. *Constructive Approximation*, 43(3):463–494, 2016.

[66] E. L. Ince. *Ordinary differential equations*. Dover, New York, 1956.

[67] N. Iorgov, O. Lisovyy, and Y. Tykhyy. Painlevé VI connection problem and monodromy of $c = 1$ conformal blocks. *Journal of High Energy Physics*, 2013(12):29, 2013.

[68] A. R. Its and A. A. Kapaev. The method od isomonodromy deformations and connection formulas for the second Painlevé transcendent. *Mathematics of the USSR-Izvestiya*, 31(1):193–207, 1988.

[69] A. R. Its, O. Lisovyy, and A. Prokhorov. Monodromy dependence and connection formulae for isomonodromic tau functions. *Duke Math. J.*, 167(7):1347–1432, 05 2018.

[70] A. R. Its, O. Lisovyy, and Y. Tykhyy. Connection Problem for the Sine-Gordon/Painlevé III Tau Function and Irregular Conformal Blocks. *International Mathematics Research Notices*, 2015(18):8903–8924, 2014.

[71] A. R. Its and V. Y. Novokshenov. *The Isomonodromic Deformation Method in the Theory of Painlevé Equations*. Springer-Verlag Berlin Heidelberg, 1986.

[72] A. R. Its and A. Prokhorov. Connection Problem for the Tau-Function of the Sine-Gordon Reduction of Painlevé-III Equation via the Riemann-Hilbert Approach. *International Mathematics Research Notices*, 2016(22):6856–6883, 2016.

[73] A. R. Its and A. Prokhorov. On Some Hamiltonian Properties of the Isomonodromic Tau Functions. *Reviews in Mathematical Physics*, 30(07):1840008, 2018.

[74] K. Iwaki, O. Marchal, and A. Saenz. Painlevé equations, topological type property and reconstruction by the topological recursion. *Journal of Geometry and Physics*, 124:16 – 54, 2018.

[75] K. Iwasaki. An Area-Preserving Action of the Modular Group on Cubic Surfaces and the Painlevé VI Equation. *Communications in Mathematical Physics*, 242(1):185–219, Nov 2003.

[76] M. Jimbo and T. Miwa. Monodromy perserving deformation of linear ordinary differential equations with rational coefficients. II. *Physica D: Nonlinear Phenomena*, 2(3):407–448, 1981.

[77] M. Jimbo, T. Miwa, Y. Môri, and M. Sato. Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent. *Physica D: Nonlinear Phenomena*, 1(1):80 – 158, 1980.

[78] M. Jimbo, T. Miwa, and K. Ueno. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients: I. General theory and τ-function. *Physica D: Nonlinear Phenomena*, 2(2):306 – 352, 1981.

[79] A. Kapaev. Global asymptotics of the second Painlevé transcendent. *Physics Letters A*, 167(4):356 – 362, 1992.

[80] H. Kawamuko, Y. Ohyama, H. Sakai, and K. Okamoto. Studies on the Painleveé equations. V. Third Painlevé equations of special type PIII(D7) and PIII(D8). *Journal of Mathematical Sciences (University of Tokio)*, pages 145–204, 2006.

[81] P. Kokocki. Total integrals of solutions for the Painlevé II equation and singularity formation in the vortex patch dynamics. arXiv:1808.09039v5 [math.CA].

[82] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin. *Quantum Inverse Scattering Method and Correlation Functions*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1993.

[83] I. V. Krasovsky. Gap probability in the spectrum of random matrices and asymptotics of polynomials orthogonal on an arc of the unit circle. *International Mathematics Research Notices*, 2004(25):1249–1272, 2004.

[84] I. M. Krichever. Isomonodromy equations on algebraic curves, canonical transformations and Whitham equations. *Moscow Mathematical Journal*, 2:717–752, 2002.

[85] O. Lisovyy. Dyson's constant for the hypergeometric kernel. In *New Trends in Quantum Integrable Systems*, pages 243–267. World Scientific, 2010.

[86] O. Lisovyy, H. Nagoya, and J. Roussillon. Irregular conformal blocks and connection formulae for Painlevé V functions. *Journal of Mathematical Physics*, 59(9):091409, 2018.

[87] O. Lisovyy and J. Roussillon. On the connection problem for Painlevé I. *Journal of Physics A: Mathematical and Theoretical*, 50(25):255202, 2017.

[88] O. Lisovyy and Y. Tykhyy. Algebraic solutions of the sixth Painlevé equation. *Journal of Geometry and Physics*, 85:124–163, 2014.

[89] B. Malgrange. Sur les déformations isomonodromiques. I. Singularités réguliìres. *Cours de l'institut Fourier*, 17:1–26, 1982.

[90] Y. I. Manin. Sixth Painlevé Equation, Universal Elliptic Curve, and Mirror of $\mathbf{P}^2$. arXiv:alg-geom/9605010.

[91] O. Marchal and N. Orantin. Isomonodromic deformations of a rational differential system and reconstruction with the topological recursion: the $\mathfrak{sl}_2$ case. arXiv:1901.04344 [math-ph].

[92] M. Mazzocco. Rational solutions of the Painlevé VI equation. *Journal of Physics A: Mathematical and General*, 34(11):2281–2294, 2001.

[93] B. M. McCoy. Connection formulas for the lambda generalized Ising correlation functions. arXiv:1902.08550 [math-ph].

[94] B. M. McCoy, C. A. Tracy, and T. T. Wu. Painlevé functions of the third kind. *Journal of Mathematical Physics*, 18(5):1058–1092, 1977.

[95] B. M. McCoy and T. T. Wu. *The Two-Dimensional Ising Model*. Harvard Univ. Press: Cambridge MAs, 1973.

[96] T. Miwa. Painlevé Property of Monodromy Preserving Deformation Equations and the Analyticity of τ Functions. *Publications of the Research Institute for Mathematical Sciences*, 17:703–721, 1981.

[97] Y. Murata. Rational solutions of the second and the fourth Painlevé equations. *Funkcialaj Ekvacioj. Serio Internacia*, 28(1):1–32, 1985.

[98] Y. Murata. Classical solutions of the third Painlevé equation. *Nagoya Mathematical Journal*, 139:37–65, 1995.

[99] J. M. Myers. Wave Scattering and the Geometry of a Strip. *Journal of Mathematical Physics*, 6(11):1839–1846, 1965.

[100] H. Nagoya. Irregular conformal blocks, with an application to the fifth and fourth Painlevé equations. *Journal of Mathematical Physics*, 56(12):123505, 2015.

[101] D. Niles. *The Riemann-Hilbert-Birkhoff inverse monodromy problem and connection formulae for the third Painlevé transcendent*. PhD thesis, IUPUI, 2009.

[102] K. Nishioka. A note on the transcendency of Painlevé's first transcendent. *Nagoya Mathematical Journal*, 109:63–67, 1988.

[103] F. Novaes and B. C. da Cunha. Isomonodromy, Painlevé transcendents and scattering off of black holes. *Journal of High Energy Physics*, 2014(7):132, 2014.

[104] K. Okamoto. Polynomial Hamiltonians associated with Painlevé equations, I. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 56(6):264–268, 1980.

[105] P. Painlevé. Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme. *Acta Mathematica*, 25:1–85, 1902.

[106] J. Palmer. Zeros of the Jimbo, Miwa, Ueno tau function. *Journal of Mathematical Physics*, 40(12):6638–6681, 1999.

[107] I. Rumanov. Hard Edge for β-Ensembles and Painlevé III. *International Mathematics Research Notices*, 2014(23):6576–6617, 2013.

[108] M. Sato, T. Miwa, and M. Jimbo. Aspects of Holonomic quantum fields. In D. Iagolnitzer, editor, *Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory*, pages 429–491, Berlin, Heidelberg, 1980. Springer Berlin Heidelberg.

[109] K. Takasaki. Painlevé-Calogero correspondence revisited. *Journal of Mathematical Physics*, 42(3):1443–1473, 2001.

[110] C. A. Tracy. Asymptotics of a τ-function arising in the two-dimensional Ising model. *Communications in Mathematical Physics*, 142(2):297–311, 1991.

[111] C. A. Tracy and H. G. Vaidya. Transverse time-dependent spin correlation functions for the one-dimensional XY model at zero temperature. *Physica A: Statistical Mechanics and its Applications*, 92(1):1 – 41, 1978.

[112] H. Umemura. On the irreducibility of the first differential equation of Painlevé. *Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata*, II:771–789, 1988.

[113] H. Umemura. Birational automorphism groups and differential equations. *Nagoya Mathematical Journal*, 119:1–80, 1990.

[114] H. Umemura. Second proof of the irreducibility of the first differential equation of Painlevé. *Nagoya Mathematical Journal*, 117:125–171, 1990.

[115] H. Umemura and H. Watanabe. Solutions of the second and fourth Painlevé equations. I. *Nagoya Mathematical Journal*, 148:151–198, 1997.

[116] H. Umemura and H. Watanabe. Solutions of the third Painlevé equation. I. *Nagoya Mathematical Journal*, 151:1–24, 1998.

[117] W. Van Assche. *Orthogonal Polynomials and Painlevé Equations*. Cambridge University Press, 2017.

[118] W. R. Wasow. *Asymptotic Expansions for Ordinary Differential Equations*. Interscience-Wiley, New-York, 1965.

[119] H. Watanabe. Solutions of the fifth Painlevé equation. I. *Hokkaido Mathematical Journal*, 24(2):231–267, 1995.

[120] H. Watanabe. Birational canonical transformations and classical solutions of the sixth Painlevé equation. *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV*, 27(3-4):379–425, 1999.

[121] H. Widom. Asymptotics of a Class of Operator Determinants. In Y. M. Erusalimsky, I. Gohberg, S. M. Grudsky, V. Rabinovich, and N. Vasilevski, editors, *Modern Operator Theory and Applications: The Igor Borisovich Simonenko Anniversary Volume*, pages 249–256. Birkhäuser Basel, Basel, 2007.

[122] D. Yamakawa. Tau functions and Hamiltonians of isomonodromic deformations. *Josai Mathematical Monographs*, 10:139–160, 2017.