Differential equations for the KPZ and periodic KPZ fixed points.

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The work was supported by NSF postdoctoral fellowship.

SIAM Conference on Nonlinear Waves and Coherent Structures.
https://arxiv.org/abs/2208.11638a

September 2nd, 2022

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KPZ fixed point

KPZ fixed point

The **KPZ fixed point**

$$\mathcal{H}^{(\mathrm{KPZ})}(\gamma,\tau), \qquad (\gamma,\tau)\in\mathbb{R}\times\mathbb{R}_+, (\mathrm{per})$$

is a 1+1 dimensional random field that is conjectured to be the universal limit for the height fluctuations of the random growth models belonging to the KPZ universality class.

If we consider random growth models in the KPZ universality class on a periodic domain (instead of the infinite line) and take a large time, large period limit in a certain critical way, then a new (1+1) dimensional random field emerges called the **periodic KPZ fixed point**.

$$\mathcal{H}^{(\mathrm{per})}(\gamma,\tau), \qquad (\gamma,\tau)\in[0,1)\times\mathbb{R}_+.$$

Initial condition

We consider the most well-studied initial condition for the growth model, the **narrow wedge initial condition** given by $\mathcal{H}(\gamma, \tau = 0) = 0$ for $\gamma = 0$ and $\mathcal{H}(\gamma, \tau = 0) = -\infty$ for $\gamma \neq 0$.

Distributions

Fix a positive integer m and consider the functions of $3m$ variables

$$\mathfrak{F}^{(\mathrm{KPZ})}(\mathrm{h},\gamma,\tau) := \mathbb{P}\left(\bigcap_{i=1}^{m}\{\mathcal{H}^{(\mathrm{KPZ})}(\gamma_i,\tau_i) \le \mathrm{h}_i\}\right), \tag{1}$$

where $\tau = (\tau_1, \cdots, \tau_m)^T \in \mathbb{R}_+^m$, $\gamma = (\gamma_1, \cdots, \gamma_m)^T \in \mathbb{R}^m$, and $\mathrm{h} = (\mathrm{h}_1, \cdots, \mathrm{h}_m)^T \in \mathbb{R}^m$ represent time, position, and height, respectively.

Similarly for $\gamma = (\gamma_1, \cdots, \gamma_m)^T \in [0,1)^m$

$$\mathfrak{F}^{(\mathrm{per})}(\mathrm{h},\gamma,\tau) := \mathbb{P}\left(\bigcap_{i=1}^{m}\{\mathcal{H}^{(\mathrm{per})}(\gamma_i,\tau_i) \le \mathrm{h}_i\}\right), \tag{2}$$

We are interested in differential equations related to $\mathfrak{F}^{(\mathrm{KPZ})}(\mathrm{h},\gamma,\tau)$ and $\mathfrak{F}^{(\mathrm{per})}(\mathrm{h},\gamma,\tau)$.

Known results on the differential equations

- $m = 1$:
  - Painlevé-II equation, observed by Tracy (1994),
  - Kadomtsev-Petviashvili-II equation, observed for periodic case by Baik, Liu, Silva (2022)
- $\tau_1 = \ldots = \tau_m$:
  - Matrix Kadomtsev-Petviashvili-II equation derived by Quastel and Remenik (2022),
  - Tracy-Widom system of ODEs (2003),
  - scalar PDE by Adler and Van Moerbeke (2005) and Wang (2009).

Cubic admissible operator

Definition

- Define the cubic exponential functions

$$\mathrm{m}_j(z) := \exp\left(\mathrm{t}_j z^3 + \mathrm{y}_j z^2 + \mathrm{x}_j z\right), \qquad z \in \mathbb{C}, \tag{3}$$

- Introduce the $(m+1) \times (m+1)$ diagonal matrix-valued function

$$\Delta(z) = \Delta(z \,|\, \mathrm{x}, \mathrm{y}, \mathrm{t}) := \mathrm{diag}\left(\mathrm{m}_1(z), \mathrm{m}_2(z), \ldots, \mathrm{m}_m(z), 1\right). \tag{4}$$

- $(m+1)$-dimensional column vector functions of the form

$$\mathrm{f}(u) = \mathrm{f}(u \,|\, \mathrm{x}, \mathrm{y}, \mathrm{t}) := \mathrm{c}(u)\Delta(u \,|\, \mathrm{x}, \mathrm{y}, \mathrm{t})\mathbf{U}(u), \tag{5}$$

$$\mathrm{g}(u) = \mathrm{g}(u \,|\, \mathrm{x}, \mathrm{y}, \mathrm{t}) := \frac{1}{\mathrm{c}(u)}\Delta(u \,|\, \mathrm{x}, \mathrm{y}, \mathrm{t})^{-1}\mathbf{V}(u) \tag{6}$$

where

- $\mathbf{U}$ and $\mathbf{V}$ are $(m+1)$-dimensional column vector functions.
- c is a non-vanishing scalar function (which may depend on $\mathrm{x}, \mathrm{y}, \mathrm{t}$);

Definition

Let $\Omega \subset \mathbb{C}$ be either a union of contours or a discrete set. μ is either counting measure or $d\mu = dz$.

Definition

We call the integral operator $\mathrm{H} : L^2(\Omega, \mu) \to L^2(\Omega, \mu)$ with the kernel

$$\mathrm{H}(u, v) = \frac{\mathrm{f}(u)^T \mathrm{g}(v)}{u - v} \tag{7}$$

cubic admissible operator if functions **U** and **V** above do not depend on $\mathrm{x}, \mathrm{y}, \mathrm{t}$. We call it **strongly cubic admissible operator** if functions **U** and **V** also do not depend on u.

Relation to KPZ and periodic KPZ fixed points

Theorem (Baik, Liu (2019), Liu (2022))

$$\mathfrak{F}^{(\mathrm{KPZ})}(\mathsf{h},\gamma,\tau)=\frac{1}{(2\pi\mathrm{i})^{m-1}}\oint\cdots\oint\mathfrak{D}^{(\mathrm{KPZ})}(\mathsf{h},\gamma,\tau\,|\,\zeta)\prod_{i=1}^{m-1}\frac{\mathrm{d}\zeta_i}{(1-\zeta_i)\zeta_i}. \tag{8}$$

$$\mathfrak{F}^{(\mathrm{per})}(\mathsf{h},\gamma,\tau):=\frac{1}{(2\pi\mathrm{i})^{m}}\oint\cdots\oint\mathfrak{C}^{(\mathrm{per})}(\zeta)\mathfrak{D}^{(\mathrm{per})}(\mathsf{h},\tau,\gamma\,|\,\zeta)\prod_{i=1}^{m}\frac{\mathrm{d}\zeta_i}{\zeta_i} \tag{9}$$

where $\mathfrak{D}^{(\mathrm{KPZ})}(\mathsf{h},\tau,\gamma\,|\,\zeta)$ *and* $\mathfrak{D}^{(\mathrm{per})}(\mathsf{h},\tau,\gamma\,|\,\zeta)$ *are Fredholm determinants.*

Relation to KPZ and periodic KPZ fixed points

Theorem (Baik, Prokhorov, Silva, 2022)

- $\mathfrak{D}^{(\mathrm{per})}(\mathsf{h}, \tau, \gamma \,|\, \zeta)$ *is Fredholm determinant of cubic admissible operator with*

$$\mathsf{t}_i = -\tau_i/3, \qquad \mathsf{y}_i = \gamma_i, \qquad \mathsf{x}_i = \mathsf{h}_i. \tag{10}$$

- $\mathfrak{D}^{(\mathrm{KPZ})}(\mathsf{h}, \gamma, \tau \,|\, \zeta)$ *is Fredholm determinant of strongly cubic admissible operator with*

$$\mathsf{t}_i = -\tau_i/3, \qquad \mathsf{y}_i = \gamma_i/2, \qquad \mathsf{x}_i = \mathsf{h}_i. \tag{11}$$

Deformation identity

Residue matrix

Definition

Let H be a cubic admissible operator on $L^2(\Omega, \mu)$ and assume that $1 - \mathrm{H}$ is invertible. Define the $(m+1) \times (m+1)$ matrix

$$\Phi_k = \Phi_k(\mathrm{x}, \mathrm{y}, \mathrm{t}) := \int_\Omega s^{k-1}((1-\mathrm{H})^{-1}\mathrm{f})(s)\mathrm{g}(s)^T \mathrm{d}\mu(s) \tag{12}$$

Introduce matrix-valued functions $\mathrm{q}, \mathrm{p}, \mathrm{r}, \mathrm{s}$ through the formula

$$\Phi_1 = \begin{pmatrix} \mathrm{q} & \mathrm{p} \\ \mathrm{r} & \mathrm{s} \end{pmatrix}, \qquad \Phi_k = \begin{pmatrix} \mathrm{q}_k & \mathrm{p}_k \\ \mathrm{r}_k & \mathrm{s}_k \end{pmatrix} \tag{13}$$

with q and s being of sizes $n \times n$ and $(m+1-n) \times (m+1-n)$, respectively.

Deformation identity

Let E_i be the $(m+1)\times(m+1)$ matrix whose (i,i) entry is 1 and all other entries are 0.

Proposition

For $i \in \{1,\cdots,m\}$ the Fredholm determinant of cubic admissible operator satisfies.

$$\partial_{\mathsf{x}_i} \log\det(1-\mathsf{H}) = -\operatorname{Tr}\left(\Phi_1 \mathsf{E}_i\right) \qquad .$$

$$\partial_{\mathsf{y}_i} \log\det(1-\mathsf{H}) = \operatorname{Tr}\left[(\Phi_1^2 - 2\Phi_2)\mathsf{E}_i\right],$$

$$\partial_{\mathsf{t}_i} \log\det(1-\mathsf{H}) = \operatorname{Tr}\left[(-\Phi_1^3 + 2\Phi_1\Phi_2 + \Phi_2\Phi_1 - 3\Phi_3)\mathsf{E}_i\right],$$

Main result: Differential equations

Differential equations for cubic admissible operators

- Multicomponent KP hierarchy
- Matrix Kadomtsev-Petviashvili-II equation
- Coupled martix NLS and mKdV systems
- Generalization of Tracy-Widom system of ODEs. (strongly cubic admissible case only)
- Scalar PDEs for $m = 2$. (strongly cubic admissible case only)

Idea of proof: dressing method for Riemann-Hilbert problems, Lax pair.

Reminder

$$\Phi_k = \Phi_k(\mathrm{x}, \mathrm{y}, \mathrm{t}) := \int_\Omega s^{k-1}((1-\mathrm{H})^{-1}\mathrm{f})(s)\mathrm{g}(s)^T \mathrm{d}\mu(s) \tag{14}$$

$$\Phi_1 = \begin{pmatrix} \mathrm{q} & \mathrm{p} \\ \mathrm{r} & \mathrm{s} \end{pmatrix}, \qquad \Phi_k = \begin{pmatrix} \mathrm{q}_k & \mathrm{p}_k \\ \mathrm{r}_k & \mathrm{s}_k \end{pmatrix} \tag{15}$$

Here q and s have sizes $n \times n$ and $(m+1-n) \times (m+1-n)$, respectively. We take $1 \leq n \leq m$.

Multicomponent KP hierarchy

Relabel the variables

$$x_1^{(i)} = \mathrm{x}_i, \qquad x_2^{(i)} = \mathrm{y}_i, \qquad x_3^{(i)} = \mathrm{t}_i, \qquad i \in \{1, \dots, m\}.$$

Define the formal $n \times n$ matrix-valued pseudo-differential operators on the variables $x_1^{(1)}, \cdots, x_1^{(n)}$,

$$\partial := \sum_{j=1}^{n} \partial_{x_1^{(j)}} \qquad \text{and} \qquad \mathrm{P} := \mathrm{I}_n + \sum_{k=1}^{\infty} \mathrm{q}_k \partial^{-k}, \tag{16}$$

and for $j \in \{1, \cdots, n\}$ and $\ell \in \{1, 2, 3\}$ also introduce

$$\mathrm{B}_\ell^{(j)} := (\mathrm{P}\mathrm{E}_j^{(n)} \partial^\ell \mathrm{P}^{-1})_+. \tag{17}$$

Here $\mathrm{E}_j^{(n)}$ is $n \times n$ matrix.

Multicomponent KP hierarchy

Proposition (Sato equations)

Let H *be a cubic admissible operator. Then the following equation holds for* $1 \leq n \leq m$*.*

$$\frac{\partial \mathrm{P}}{\partial x_\ell^{(i)}} = \mathrm{B}_\ell^{(i)}\mathrm{P} - \mathrm{P}\mathrm{E}_i^{(n)}\partial^\ell \qquad \text{for } i \in \{1, \cdots, n\} \text{ and } \ell \in \{1, 2, 3\}. \tag{18}$$

Matrix Kadomtsev-Petviashvili-II equation

Introduce the differential operators

$$\partial_{\mathrm{t}} = \sum_{k=1}^{n} \partial_{\mathrm{t}_k}, \qquad \partial_{\mathrm{y}} = \sum_{k=1}^{n} \partial_{\mathrm{y}_k} \qquad \text{and} \qquad \partial_{\mathrm{x}} = \sum_{k=1}^{n} \partial_{\mathrm{x}_k}. \tag{19}$$

Theorem

Let H *be a cubic admissible operator. For* $1 \leq n \leq m$ *we have the following result.*

ⓐ *The* $n \times n$ *matrices* $\mathrm{u} := \mathrm{pr}$ *and* q *satisfy the matrix KP-II equation*

$$-4\partial_{\mathrm{t}}\mathrm{u}+\partial_{\mathrm{x}}^3\mathrm{u}+6\partial_{\mathrm{x}}(\mathrm{u}^2)-3\partial_{\mathrm{y}}^2\mathrm{q}+6[\mathrm{u},\partial_{\mathrm{y}}\mathrm{q}] = 0, \qquad \partial_{\mathrm{x}}\mathrm{q} = -\mathrm{u}. \tag{20}$$

ⓑ *The* $(m+1-n) \times (m+1-n)$ *matrices* $\mathrm{v} := \mathrm{rp}$ *and* s *satisfy the matrix KP-II equation*

$$-4\partial_{\mathrm{t}}\mathrm{v}+\partial_{\mathrm{x}}^3\mathrm{v}+6\partial_{\mathrm{x}}(\mathrm{v}^2)+3\partial_{\mathrm{y}}^2\mathrm{s}+6[\mathrm{v},\partial_{\mathrm{y}}\mathrm{s}] = 0, \qquad \partial_{\mathrm{x}}\mathrm{s} = \mathrm{v}. \tag{21}$$

Coupled martix NLS and mKdV systems

Theorem (Coupled matrix NLS with complex time and mKdV)

Let H *be a cubic admissible operator. For* $1 \leq n \leq m$ *we have the following result.*

- (a) *As functions of* x *and* y*, the matrices* p *and* r *satisfy a system of coupled matrix nonlinear Schrödinger (NLS) equations with complex time* $\mathrm{y} \mapsto \mathrm{iy}$,

$$\partial_{\mathrm{y}}\mathrm{p} = \partial_{\mathrm{x}}^2\mathrm{p} + 2\mathrm{prp}, \qquad \partial_{\mathrm{y}}\mathrm{r} = -\partial_{\mathrm{x}}^2\mathrm{r} - 2\mathrm{rpr}. \tag{22}$$

- (b) *As functions of* t *and* x*, the matrices* p *and* r *satisfy a system of coupled matrix modified Korteweg-de Vries (mKdV) equations*

$$\partial_{\mathrm{t}}\mathrm{p} = \partial_{\mathrm{x}}^3\mathrm{p} + 3(\partial_{\mathrm{x}}\mathrm{p})\mathrm{rp} + 3\mathrm{pr}(\partial_{\mathrm{x}}\mathrm{p}), \qquad \partial_{\mathrm{t}}\mathrm{r} = \partial_{\mathrm{x}}^3\mathrm{r} + 3(\partial_{\mathrm{x}}\mathrm{r})\mathrm{pr} + 3\mathrm{rp}(\partial_{\mathrm{x}}\mathrm{r}). \tag{23}$$

System of ODEs

Define $(m+1)\times(m+1)$ matrices

$$\mathrm{M}_{\mathrm{t}} := \mathrm{diag}(\mathrm{t}_1, \cdots, \mathrm{t}_m, 0), \quad \mathrm{M}_{\mathrm{y}} := \mathrm{diag}(\mathrm{y}_1, \cdots, \mathrm{y}_m, 0),$$
$$\mathrm{M}_{\mathrm{x}} := \mathrm{diag}(\mathrm{x}_1, \cdots, \mathrm{x}_m, 0),$$

and the differential operator

$$\partial := \sum_{k=1}^{m} \mathrm{t}_k \partial_{\mathrm{x}_k}. \tag{24}$$

If we change variable $\mathrm{x}_k \to \mathrm{x}_k + \xi \mathrm{t}_k$, then $\partial = \frac{d}{d\xi}$.

Theorem

Let H *be a* <u>*strongly*</u> *cubic integrable operator. Then the following holds:*

$$3\partial^2 \Phi_1 = 2[\partial\Phi_1, \mathrm{M}_{\mathrm{y}}] + [[\Phi_1, \mathrm{M}_{\mathrm{t}}], 3\partial\Phi_1 - 2[\Phi_1, \mathrm{M}_{\mathrm{y}}] - \mathrm{M}_{\mathrm{x}}]. \tag{25}$$

Scalar PDEs for $m = 2$

Introduce notation $M = \det(1 - \mathrm{H})$ with H - strongly cubic admissible operator and introduce new variables

$$
\begin{aligned}
&\mathrm{t}_1 = -1/3, \qquad \mathrm{t}_2 = -\frac{t}{3} \qquad , \mathrm{y}_1 = 0, \qquad \mathrm{y}_2 = y, \\
&\mathrm{x}_1 = \frac{E+W}{2}, \qquad \mathrm{x}_2 = \frac{E-W}{2} - \frac{y^2}{t} \qquad .
\end{aligned} \tag{26}
$$

We obtained using symbolic computations the following four equations

$$
\begin{aligned}
&24(1-t)t^2\partial_E\partial_t M + 16t^2\partial_E^4 M + 96t^2(\partial_E^2 M)^2 + 3(1-t)^2 t^2 \partial_y^2 M \\
&+ 4yt(3-7t)\partial_E\partial_y M + 12yt(t-1)\partial_y\partial_W M - 4(2Et^2 + 3y^2)\partial_E^2 M \\
&- 8Wt^2\partial_E\partial_W M + 12y^2\partial_W^2 M + 2t(3-t)\partial_E M + 6t(t-1)\partial_W M = 0.
\end{aligned} \tag{27}
$$

Scalar PDEs

$$\begin{aligned}
&3(t-1)\partial_E^4 M - 8(t+1)\partial_E^3\partial_W M + 6(t-1)\partial_E^2\partial_W^2 M + (1-t)\partial_W^4 M \\
&+ 12W\partial_E^2 M + 8E\partial_E\partial_W M - 4W\partial_W^2 M + 3t(1-t)\partial_y^2 M - 8y\partial_W\partial_y M \\
&- 4\partial_W M + 18(t-1)(\partial_E^2 M)^2 + 24(t-1)(\partial_E\partial_W M)^2 \\
&+ 6(1-t)(\partial_W^2 M)^2 + 12(t-1)\partial_E^2 M\partial_W^2 M \\
&- 48(t+1)\partial_E^2 M\partial_E\partial_W M = 0.
\end{aligned} \tag{28}$$

Scalar PDEs

$$
\begin{aligned}
&\tau(\tau+1)(19\tau+21)\partial_E^4 M + 60\tau(1-\tau^2)\partial_E^3\partial_W M \\
&+ 6(\tau-1)(11\tau-9)\partial_E^2\partial_W^2 M - 4(\tau+1)(7\tau-3)\partial_E\partial_W^3 M \\
&+ 3\tau(\tau^2-1)\partial_W^4 M - 12(E\tau^2-4W\tau^2+E\tau+6W\tau+5y^2)\partial_E^2 M \\
&+ 12\tau(5E\tau-W\tau-5E-W)\partial_E\partial_W M + 12(W\tau^2+W\tau+5y^2)\partial_W^2 M \\
&+ 6\tau^2(1-\tau)(3\tau-2)\partial_y^2 M + 48\tau^2(\tau+1)\partial_E\partial_\tau M + 18\tau(\tau+1)\partial_E M \\
&- 6\tau(7\tau-3)\partial_W M + 6(\tau+1)(19\tau+21)(\partial_E^2 M)^2 + 18\tau(\tau^2-1)(\partial_W^2 M) \\
&+ 24\tau(\tau-1)(11\tau-9)(\partial_E\partial_W M)^2 + 12(\tau-1)(11\tau-9)\partial_E^2 M\partial_W^2 M \\
&+ 360\tau(1-\tau^2)\partial_E^2 M\partial_E\partial_W M + 12\tau(3-7\tau)(1+\tau)\partial_W^2 M\partial_E\partial_W M = 0
\end{aligned}
\tag{29}
$$

Scalar PDEs

$$
\begin{aligned}
&3(\tau+1)(3\tau-7)\partial_E^4 M - 20\tau(\tau+1)(\tau+3)\partial_E^3\partial_W M \\
&+6\tau(\tau^2-9)\partial_E\partial_W M + 12\tau(\tau^2-1)\partial_E\partial_W^3 M(\tau+1)(3-7\tau)\partial_W^4 M \\
&+12(E\tau^2+4W\tau^2+E\tau+6W\tau+y^2)\partial_E^2 M \\
&+12\tau(3E\tau+W\tau+5E+W)\partial_E\partial_W M - 12(W\tau^2+W\tau+y^2)\partial_W^2 M \\
&-6\tau^2(3\tau^2+3\tau-2)\partial_{\mathrm{y}}^2 M + 48\tau^2(\tau+1)\partial_W\partial_\tau M - 18\tau(\tau+1)\partial_E M \\
&-6\tau(\tau+3)\partial_W M + 18(\tau+1)(3\tau-7)(\partial_E^2 M)^2 \\
&+6\tau(\tau+1)(3-7\tau)(\partial_W^2 M)^2 + 24\tau(\tau^2-9)(\partial_E\partial_W M)^2 \\
&+12\tau(\tau^2-9)\partial_E^2 M\partial_W^2 M - 120\tau(\tau+1)(\tau+3)\partial_E^2 M\partial_E\partial_W M \\
&+72\tau(\tau^2-1)\partial_W^2 M\partial_E\partial_W M = 0
\end{aligned}
\tag{30}
$$

Interesting problems

- Prove equations of multicomponent KP hierarchy for general initial condition.
- Obtain difference equations for growth processes before KPZ limit as discretization of multicomponent KP hierarchy.
- Asymptotic analysis of distributions. There are interesting questions even for the case of Airy2 process corresponding to $\tau_1 = \ldots = \tau_m$.

Thank you!