Nonlinear steepest descent on a torus: A case study of the Landau-Lifshitz equation.

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8. The talk is based on [DIP25].

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- We assume the boundary condition

$$S \to (0,0,1), \quad x \to \pm\infty$$

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$$\frac{\partial S}{\partial t} = S \times \frac{\partial^2 S}{\partial x^2} + S \times JS$$

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- Landau-Lifshitz model admits degenerations to the Sine-Gordon equation when $J_1 \to 0$ and to the NLS equation when $J_1, J_2 \to 0$.

Lax pair

- The Lax pair is given by

$$\begin{cases} \dfrac{\partial \Psi}{\partial x} = U\Psi \\ \dfrac{\partial \Psi}{\partial t} = V\Psi \end{cases}, \quad \text{where}$$

$$U = -\mathrm{i} \sum_{j=1}^{3} \sigma_j S_j w_j, \quad V = \mathrm{i} \sum_{\substack{j,m,n=1 \\ j \neq m \neq n}}^{3} \sigma_j S_j w_m w_n + \mathrm{i} \sum_{j=1}^{3} \sigma_j P_j w_j$$

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σ_j are Pauli matrices and parameters w_j solve algebraic system

$$w_i^2 - w_j^2 = \frac{1}{4}(J_j - J_i); \qquad i, j = 1, 2, 3$$

Elliptic parametrization

- The parameters w_j admit parametrization using elliptic functions

$$w_1 = \rho \frac{1}{\mathrm{sn}(\lambda, k)}, \quad w_2 = \rho \frac{\mathrm{dn}(\lambda, k)}{\mathrm{sn}(\lambda, k)}, \quad w_3 = \rho \frac{\mathrm{cn}(\lambda, k)}{\mathrm{sn}(\lambda, k)}.$$

$$\rho = \frac{\sqrt{J_3 - J_1}}{2}, \quad k = \sqrt{\frac{J_2 - J_1}{J_3 - J_1}}$$ – elliptic modulus.

where

$$\lambda \in \mathbb{T}^2 = \{\lambda : |\mathrm{Re}(\lambda)| \leq 2K, |\mathrm{Im}(\lambda)| \leq 2K'\}$$

and K, K' are complete elliptic integrals.

Jost solution

- We assume "fast decaying" and smooth initial conditions

$$S|_{t=0} - (0,0,1) \in \mathcal{S}(\mathbb{R})$$

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- We define Jost solutions that satisfy the differential equation

$$\frac{\partial F_\pm}{\partial x}(\lambda, x) = U(\lambda, x)F_\pm(\lambda, x), \quad U(\lambda, x) = -\mathrm{i}\sum_{j=1}^{3} \sigma_j S_j(x) w_j(\lambda)$$

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and the boundary conditions

$$F_{\pm}(\lambda, x) \sim \mathrm{e}^{-\mathrm{i}w_3(\lambda)x\sigma_3}, \quad , x \to \pm\infty$$

Symmetries

- The following shift properties hold

$$
\begin{aligned}
&w_1(\lambda + 2K) = -w_1(\lambda), \quad w_1(\lambda + 2\mathrm{i}K') = w_1(\lambda),\\
&w_2(\lambda + 2K) = -w_2(\lambda), \quad w_2(\lambda + 2\mathrm{i}K') = -w_2(\lambda),\\
&w_3(\lambda + 2K) = w_3(\lambda), \quad w_3(\lambda + 2\mathrm{i}K') = -w_3(\lambda),
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- They imply the symmetries of coefficient matrix

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\sigma_3 U(\lambda+2K,x)\sigma_3=U(\lambda,x), \quad \sigma_1 U(\lambda+2\mathrm{i}K',x)\sigma_1=U(\lambda,x).
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- and the symmetries of Jost solutions

$$\sigma_3 F_\pm(\lambda+2K,x)\sigma_3=F_\pm(\lambda,x),\quad \sigma_1 F_\pm(\lambda+2\mathrm{i}K',x)\sigma_1=F_\mp(\lambda,x).$$

Scattering data

- By taking the ratio of Jost solutions we define the scattering matrix

$$S(\lambda) = (F_-(\lambda, x))^{-1} F_+(\lambda, x) = \begin{pmatrix} \mathrm{a}(\lambda) & -\overline{\mathrm{b}(\bar{\lambda})} \\ \mathrm{b}(\lambda) & \overline{\mathrm{a}(\bar{\lambda})} \end{pmatrix}.$$

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$$\mathrm{r}(\lambda) = \frac{\mathrm{b}(\lambda)}{\mathrm{a}(\lambda)}.$$

Scattering data

- By taking the ratio of Jost solutions we define the scattering matrix

$$S(\lambda) = \left(F_-(\lambda, x)\right)^{-1} F_+(\lambda, x) = \begin{pmatrix} a(\lambda) & -\overline{b(\bar{\lambda})} \\ b(\lambda) & \overline{a(\bar{\lambda})} \end{pmatrix}.$$

- The reflection coefficient is given by

$$r(\lambda) = \frac{b(\lambda)}{a(\lambda)}.$$

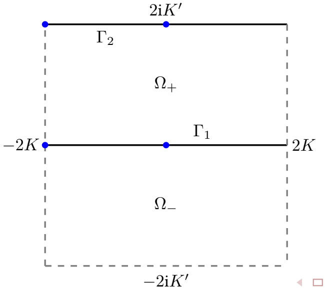
- We make the following transformation

$$\Upsilon_\pm(\lambda, x) = F_\pm(\lambda, x) e^{ixw_3(\lambda)\sigma_3} = \left(\upsilon_\pm^{(1)}(\lambda, x), \upsilon_\pm^{(2)}(\lambda, x)\right).$$

Analytical properties of Jost solutions

- Let us denote

$$\Omega_+ = \left\{\lambda : 0 \le \mathrm{Im}(\lambda) \le 2K'; |\mathrm{Re}\lambda| \le 2K\right\},$$
$$\Omega_- = \left\{\lambda : -2K' \le \mathrm{Im}(\lambda) \le 0; |\mathrm{Re}\lambda| \le 2K\right\},$$
$$\Gamma_1 = \{\lambda \in \mathbb{T}^2 : \mathrm{Im}(\lambda) = 0\}$$
$$\Gamma_2 = \{\lambda \in \mathbb{T}^2 : \mathrm{Im}(\lambda) = 2K'\}$$



Analytical properties of Jost solutions

Lemma 1 ([Rod84])

The functions $v_\pm^{(1)}(\lambda, x)$, $v_\mp^{(2)}(\lambda, x)$ *are analytic in the domains* $\Omega_\pm$ *and bounded in the domains* $\overline{\Omega}_\pm$ *respectively. In addition* $v_\pm^{(1)}(\lambda, x), v_\pm^{(2)}(\lambda, x) \in C^\infty(\Gamma_1)$.

Ideas of the proof:

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- The function $\mathrm{Im}(w_3(\lambda))$ does not change sign in $\Omega_{\pm}$

$$-\infty < \mathrm{Im}(w_3(\lambda)) \le 0 \quad \text{for} \quad \lambda \in \Omega_+;$$
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- Singularity of $w_3(\lambda)$ at $\lambda = 0$ needs to be taken care of separately. It is done using AKNS equation approximation for such λ

Convenient formulas for scattering data

Lemma 2

The functions a(λ), b(λ) *admit two alternative expressions.*

1. *They can be written as the integrals*

$$\begin{pmatrix} \mathrm{a}(\lambda) \\ \mathrm{b}(\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(\mathbb{1}-\sigma_3)w_3(\lambda)\tau} \left(U(\lambda,\tau) + \mathrm{i}w_3(\lambda)\sigma_3\right)$$

$$\times\, \upsilon_-^{(1)}(\lambda,\tau)d\tau.$$

2. *They can also be expressed as the following determinants*

$$\mathrm{a}(\lambda) = \det(\upsilon_+^{(1)}(\lambda,x), \upsilon_-^{(2)}(\lambda,x)),$$
$$\mathrm{b}(\lambda) = \mathrm{e}^{2\mathrm{i}w_3(\lambda)x}\det(\upsilon_-^{(1)}(\lambda,x), \upsilon_+^{(1)}(\lambda,x)).$$

Reflection coefficient properties

Lemma 3

Under the soliton free assumption, $\mathrm{a}(\lambda) \neq 0$ *the reflection coefficient* $\mathrm{r}(\lambda)$ *corresponding to the initial data from the Schwartz class satisfies*

1. $\mathrm{r}(\lambda) \in C^{\infty}(\Gamma_1 \cup \Gamma_2)$
2. $\mathrm{r}(\lambda + 2K) = -\mathrm{r}(\lambda)$
3. $\mathrm{r}(\lambda + 2\mathrm{i}K') = -\overline{\mathrm{r}(\bar{\lambda})}$
4. $\mathrm{r}(0) = 0$
5. $\frac{d^n \mathrm{r}(\lambda)}{d\lambda^n} = \mathcal{O}(\lambda^m), \quad \lambda \to 0, \quad \forall n, m \in \mathbb{N}.$

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- Symmetry properties follow from symmetry properties of Jost solutions.
- Smoothness follows from the second representation in terms of determinant.
- Behavior at zero is derived through the asymptotic analysis of the first representation in terms of integral. AKNS approximation near $\lambda = 0$ is needed again.

Riemann-Hilbert problem for the initial condition

- Introduce the following functions

$$Y_+(\lambda, x) = \left(\frac{\upsilon_+^{(1)}(\lambda, x)}{a(\lambda)}, \upsilon_-^{(2)}(\lambda, x) \right),$$

$$Y_-(\lambda, x) = \left(\upsilon_-^{(1)}(\lambda, x), \frac{\upsilon_+^{(2)}(\lambda, x)}{\overline{a(\bar{\lambda})}} \right),$$

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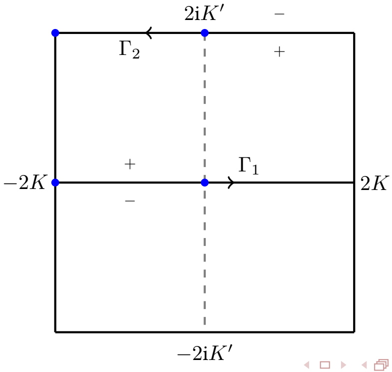
- Define the piecewise analytic function

$$Y(\lambda, x) = \begin{cases} Y_+(\lambda, x), & \lambda \in \Omega_+, \\ Y_-(\lambda, x), & \lambda \in \Omega_-. \end{cases}$$

Riemann-Hilbert problem for the initial condition

Riemann-Hilbert problem 1.

1. The function $Y(\lambda, x)$ is bounded, doubly periodic, and piecewise analytic for $\lambda \in \mathbb{T}^2/(\Gamma_1 \cup \Gamma_2)$. Orientation of the contours Γ_1, Γ_2 is as specified below.



Riemann-Hilbert problem for the initial condition

2. For $\lambda \in \Gamma_1, \Gamma_2$, the following jump condition holds

$$Y_+(\lambda, x) = Y_-(\lambda, x) G(\lambda, x),$$
$$G(\lambda, x) = \begin{pmatrix} 1 + |\mathrm{r}(\lambda)|^2 & \overline{\mathrm{r}(\lambda)} \mathrm{e}^{-2\mathrm{i}x w_3(\lambda)} \\ \mathrm{r}(\lambda) \mathrm{e}^{2\mathrm{i}x w_3(\lambda)} & 1 \end{pmatrix}.$$

3. The function $Y(\lambda, x)$ satisfies the following symmetry conditions

$$\sigma_3 Y(\lambda + 2K, x) \sigma_3 = Y(\lambda, x), \quad \sigma_1 Y(\lambda + 2\mathrm{i}K', x) \sigma_1 = Y(\lambda, x).$$

4. Function $Y(\lambda, x)$ satisfies normalization condition

$$\det(Y(\lambda, x)) = 1$$

Uniqueness

- The problem is not normalized but the symmetries and determinant conditions still fix the solution up to a sign.

Lemma 1

Solution $Y(\lambda, x)$ of the RHP 1 is unique up to a sign.

Riemann-Hilbert problem on the Riemann surfaces

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  - $C(\mu, \lambda)$ with respect to μ has pole at λ, at some point a, and zero at some divisor $\mathcal{D}$.
- The Cauchy transform

$$\int_{\Gamma} f(\mu) C(\mu, \lambda) d\mu$$

has poles at the divisor $\mathcal{D}$ with respect λ.

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- For the case of torus we take $a = K + \mathrm{i}K'$ and $\mathcal{D} = \mathrm{i}K'$.

$$C(\mu,\lambda) = \zeta(\mu-\lambda) - \zeta(\mu - \mathrm{i}K') + \zeta(\lambda - K - \mathrm{i}K') + \zeta(K),$$

where $\zeta(.)$ is the Weierstrass ζ-function.

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- The periodicity properties:

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- Poles at $\mu = \lambda$, $\mu = \mathrm{i}K'$, $\lambda = K + \mathrm{i}K'$ with residues $1, -1$, and 1 respectively.
- Zeros for $\lambda = \mathrm{i}K'$, $\mu = K + \mathrm{i}K'$.

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- Finite gap solutions [KT12, MLT12]
- Parametrix construction [DIZ97, Kor04]

Initial conditions

Theorem 2

Let $Y(\lambda, x)$ be the solution of the Riemann-Hilbert problem 1 with $\mathrm{r}(\lambda)$ *satisfying properties* (1)-(5)*. Then the function $S(x)$ constructed from it by formula*

$$Y(0,x)\sigma_3\left(Y(0,x)\right)^{-1} = \sum_{j=1}^{3} S_j(x)\sigma_j,$$

belongs to the Schwartz class: $S_1(x), S_2(x), S_3(x) - 1 \in \mathcal{S}(\mathbb{R})$, and it defines the initial data for the LL equation whose reflection coefficient is given by $\mathrm{r}(\lambda)$*.*

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- After deformation of the contour the jump is close to identity.

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- Reflection coefficient is split in the sum of analytic term and term satisfying decay properties using Fourier transform.
- After deformation of the contour the jump is close to identity.
- Consider singular integral equation

$$\chi(\lambda,x)=\mathbb{1}+\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_1\cup\Gamma_2}\chi(\mu,x)\left(G(\mu,x)-\mathbb{1}\right)C(\mu,\lambda-\mathrm{i}0)d\mu.$$

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$$\chi(\lambda, x) = \mathbb{1} + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1 \cup \Gamma_2} \chi(\mu, x) \left(G(\mu, x) - \mathbb{1} \right) C(\mu, \lambda - \mathrm{i}0) d\mu.$$

- Nonsymmetric solution of Riemann-Hilbert problem normalized by $\Phi(\mathrm{i}K')$ is given by

$$\Phi(\lambda, x) = \mathbb{1} + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1 \cup \Gamma_2} \chi(\mu, x) \left(G(\mu, x) - \mathbb{1} \right) C(\mu, \lambda) d\mu.$$

Ideas of proof

- Symmetric solution is constructed by

$$
\begin{aligned}
Y(\lambda) =& \frac{1}{\sqrt{c}}\left(\Phi(\lambda) + \sigma_3\Phi(\lambda+2K)\sigma_3 + \sigma_1\Phi(\lambda+2\mathrm{i}K')\sigma_1\right. \\
&\left.+\sigma_2\Phi(\lambda+2K+2\mathrm{i}K')\sigma_2\right), \\
c =& \det\left(\Phi(\lambda) + \sigma_3\Phi(\lambda+2K)\sigma_3 + \sigma_1\Phi(\lambda+2\mathrm{i}K')\sigma_1 +\right. \\
&\left.\sigma_2\Phi(\lambda+2K+2\mathrm{i}K')\sigma_2\right).
\end{aligned}
$$

- Problem: singularity at $\lambda = K + \mathrm{i}K'$. It is crucial for us to use the result of [Rod89] which claims that this singularity is absent already for $\Phi(\lambda, x)$ based on perturbation approach.

Time evolution

Lemma 3 ([Skl79])

Assume $t > 0$ is fixed and

$$S_1(x,t), S_2(x,t), S_3(x,t) - 1 \in \mathcal{S}(\mathbb{R})$$

are solutions of LL equation. Denote by $F_\pm(\lambda, x, t)$ the Jost solutions corresponding to $S(x,t)$ with $t > 0$. Then the matrix-valued functions

$$J_\pm(\lambda, x, t) = F_\pm(\lambda, x, t) \mathrm{e}^{2\mathrm{i}t w_1(\lambda) w_2(\lambda) \sigma_3}$$

solves equations of Lax pair of LL equation. Moreover the scattering data depends on time as

$$\mathrm{a}(\lambda, t) = \mathrm{a}(\lambda), \quad \mathrm{b}(\lambda, t) = \mathrm{b}(\lambda) \mathrm{e}^{-4\mathrm{i}t w_1(\lambda) w_2(\lambda)}.$$

Riemann-Hilbert problem corresponding to LL dynamics

Riemann-Hilbert problem 2.

1. The function $Y(\lambda, x, t)$ is bounded and piecewise analytic for $\lambda \in \mathbb{T}^2/(\Gamma_1 \cup \Gamma_2)$.
2. For $\lambda \in \Gamma_1, \Gamma_2$, the following jump condition holds

$$Y_+(\lambda, x, t) = Y_-(\lambda, x, t) G(\lambda, x, t)$$

$$G(\lambda, x, t) = \begin{pmatrix} 1 + |\mathrm{r}(\lambda)|^2 & \overline{\mathrm{r}(\lambda)} \mathrm{e}^{-2\mathrm{i}tp(\lambda,\varkappa)} \\ \mathrm{r}(\lambda)\mathrm{e}^{2\mathrm{i}tp(\lambda,\varkappa)} & 1 \end{pmatrix}.$$

where

$$p(\lambda, \varkappa) = \varkappa w_3(\lambda) - 2w_1(\lambda)w_2(\lambda), \quad \varkappa = \frac{x}{t}.$$

Riemann-Hilbert problem corresponding to LL dynamics

3. The function $Y(\lambda, x, t)$ satisfies the following symmetry conditions

$$\sigma_3 Y(\lambda + 2K, x, t)\sigma_3 = Y(\lambda, x, t),$$
$$\sigma_1 Y(\lambda + 2\mathrm{i}K', x, t)\sigma_1 = Y(\lambda, x, t).$$

4. Function $Y(\lambda, x, t)$ satisfies normalization condition

$$\det(Y(\lambda, x, t)) = 1.$$

Solution of LL from RHP

Lemma 4

Given $Y(\lambda, x, t)$ is a solution of the Riemann-Hilbert problem 2 the function

$$\Psi(\lambda, x, t) = Y(\lambda, x, t)\mathrm{e}^{-\mathrm{i}tp(\lambda,\varkappa)\sigma_3}$$

solves Lax pair of LL equation with $S_j(x, t)$ given by

$$Y(0, x, t)\sigma_3 \left(Y(0, x, t)\right)^{-1} = \sum_{j=1}^{3} S_j(x, t)\sigma_j. \qquad (1)$$

Main result

Theorem 5 ([DIP25])

Let $Y(\lambda, x, t)$ be the solution of the Riemann-Hilbert problem 2 with $\mathrm{r}(\lambda)$ *satisfying the properties* (1)-(5) *and let the vector function $S(x,t)$ be determined by equation* (1). *Then, $S(x,t)$ solves the Cauchy problem for the LL equation characterized by the reflection coefficient* $\mathrm{r}(\lambda)$ *and for* $t \to \infty, \quad 0 < m \leq \frac{x}{t} \leq M$,

$$S_1(x,t) = \frac{1}{\rho}\left(\frac{2\nu}{t\varphi_0}\right)^{1/2} w_2(\lambda_0)\cos\theta(x,t) + \mathcal{O}(t^{-\frac{2}{3}}),$$

$$S_2(x,t) = \frac{1}{\rho}\left(\frac{2\nu}{t\varphi_0}\right)^{1/2} w_1(\lambda_0)\sin\theta(x,t) + \mathcal{O}(t^{-\frac{2}{3}}),$$

$$S_3(x,t) = 1 - \frac{1}{2}\left(S_1^2(x,t) + S_2^2(x,t)\right) + \mathcal{O}(t^{-\frac{7}{6}}),$$

Parameters

where

$$\theta(x,t) = 2tp(\lambda_0,\varkappa) + \nu \log t - \frac{\pi}{4} - \arg\Gamma(\mathrm{i}\nu) + \arg \mathrm{r}_0 - 2\mathrm{c}_0 + \nu \log\left(\frac{2\varphi_0}{\beta_0^2}\right),$$

$$p(\lambda,\varkappa) = \varkappa\, w_3(\lambda) - 2w_1(\lambda)w_2(\lambda), \quad \varkappa = \frac{x}{t}$$

and the value of the stationary point $\lambda_0 \in [-2K, 0]$ is determined by the equation $\partial_\lambda p(\lambda_0,\varkappa) = 0$. With such λ_0, the parameter $\varphi_0 = -\partial_\lambda^2 p(\lambda_0,\varkappa)$ is obtained from

$$\varphi_0 = \frac{1}{\rho^2}\left(8w_1(\lambda_0)w_2(\lambda_0)w_3^2(\lambda_0) + (w_1^2(\lambda_0) + w_2^2(\lambda_0))(2w_1(\lambda_0)w_2(\lambda_0) - \varkappa w_3(\lambda_0))\right),$$

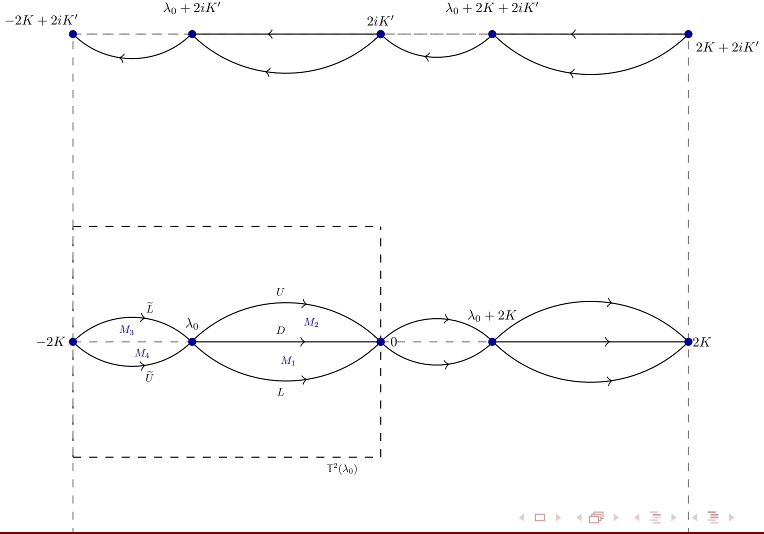
Parameters

- The remaining terms are determined as follows:

$$
\begin{aligned}
&\mathrm{r}_0 = \mathrm{r}(\lambda_0), \qquad \nu = \frac{1}{2\pi}\log(1+|\mathrm{r}_0|^2).\\
&\beta(\lambda) = \frac{\sigma(\lambda)\sigma(\lambda-2K)}{\sigma(\lambda+2\mathrm{i}K')\sigma(\lambda-2\mathrm{i}K'-2K)},\\
&\beta_0 := \frac{\sigma(-2K)}{\sigma(2\mathrm{i}K')\sigma(-2\mathrm{i}K'-2K)},\\
&\mathrm{c}_0 = \frac{1}{2\pi}\int_{\lambda_0}^{0} d\left(\log\left(1+|\mathrm{r}(\eta)|^2\right)\right)\log\beta(\eta-\lambda_0),
\end{aligned}
$$

where $\sigma(\lambda)$ denotes the Weierstrass sigma function.

Ideas of proof



Interesting questions

- Soliton gas analysis.

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- Soliton gas analysis.
- Rogue waves of infinite order.
- Zero dispersion limit.
- Approximation theory on the torus with the goal to optimize numerical computation of singular integrals of type

$$\int_0^{2K} f(\mu) C(\mu, \lambda - \mathrm{i}0) d\mu$$

for

$$C(\mu, \lambda) = \zeta(\mu - \lambda) - \zeta(\mu - \mathrm{i}K') + \zeta(\lambda - K - \mathrm{i}K') + \zeta(K),$$

and $\zeta(.)$ is the Weierstrass ζ-function.

1. Landau-Lifshitz equation

2. Riemann-Hilbert approach

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Thank You