Nonlinear steepest descent on a torus: A case study of the Landau-Lifshitz equation.

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Inverse scattering method

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- The major breakthrough was the work of Gardner, Greene, Kruscal and Miura. They showed that the nonlinear Korteweg–De Vries (KdV) dynamic for the potential of Schrödinger equation transforms to linear dynamic for the scattering data.

Other integrable models

- In the later years many more models fitting to the inverse scattering framework were discovered (see [FT07]):
  - Nonlinear Schrödinger equation
  - Sine-Gordon equation
  - Continuous Heisenberg magnet.
  - Landau-Lifshitz model.
  - Toda equation
  - Boussinesq equation
  - Kadomtsev-Petviashvili equation
  - ...

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- The focus of our work is to implement this analysis for the case of Landau-Lifshitz equation.
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- The difficulty here is the setup of the problem: usually Riemann-Hilbert problems are concerned with analytic functions on Riemann sphere as the starting point, but in our case we have to start from the torus.
- In the next slides we will talk about the Heisenberg magnet (HM) model which is a simplified version of Landau-Lifshitz (LL) model.

Classical Heisenberg magnetism model

- The classical model for magnetism was introduced by Heisenberg. It is described by the following Hamiltonian

$$H = -\sum_{k\sim j} S^{(k)} \cdot S^{(j)}$$

where the classical spin vectors $S^{(k)} \in \mathbb{S}^2$ are located at some lattice sites k and the sum $\sum_{k\sim j}$ runs over nearest-neighbor pairs in the lattice.

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- The interpretation of the Hamiltonian is the energy of the system of classical spins. They tend to align with each other to minimize the energy.

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where the convergence is sufficiently fast.
- The Hamiltonian needs to be regularized and it takes form

$$H = -2 \sum_{n \in \mathbb{Z}} \log \left(\frac{1 + S^{(n)} \cdot S^{(n+1)}}{2} \right)$$

Magnetic waves

- It is possible to consider the dynamics of classical spins after introduction of convenient Poisson bracket

$$\{S_a^{(n)}, S_b^{(m)}\} = -\varepsilon_{abc}\delta_{mn}S_c^{(n)}$$

where δ_{nm} is the Kronecker symbol and ε_{abc} is the Levi-Cevita symbol.

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- More explicitly it takes form

$$\frac{\partial S^{(n)}}{\partial t} = 2S^{(n)} \times \left(\frac{S^{(n+1)}}{1 + S^{(n)} \cdot S^{(n+1)}} + \frac{S^{(n-1)}}{1 + S^{(n)} \cdot S^{(n-1)}} \right),$$

where $\times$ denotes the vector product.

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- As it was mentioned earlier, this nonlinear system of PDEs is amenable to the inverse scattering procedure, see [FT07].
- The effective indicator of the possibility of an inverse scattering transform for the nonlinear PDE is the presence of Lax pair representation. It can be interpreted as a linearization of the equation.

Lax pair for the continuous Heisenberg magnet (HM)

- To describe the Lax pair we need to introduce

$$L=\sum_{j=1}^{3} S_j\sigma_j$$

where the Pauli matrices are given by

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

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- The evolution in terms of matrix L is given by

$$\frac{\partial L}{\partial t} = \frac{1}{2\mathrm{i}}\left[L, \frac{\partial^2 L}{\partial x^2}\right]$$

$$L \to \sigma_3, \quad \text{as} \quad x \to \pm\infty$$

Lax pair for the continuous Heisenberg magnet (HM)

- The Lax pair for the HM model is given by linear systems of ODEs

$$\begin{cases} \dfrac{\partial \Phi}{\partial x} = \mathrm{i}\lambda L \Phi \\ \dfrac{\partial \Phi}{\partial t} = \left(2\mathrm{i}\lambda^2 L + \lambda L \dfrac{\partial L}{\partial x}\right) \Phi \end{cases}, \tag{1}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter.

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- The time evolution (1) of matrix L is the compatibility condition (zero curvature) condition associated to the Lax pair (1) under the assumption $L^2 = \mathbb{1}$.

$$\frac{\partial}{\partial t}(\mathrm{i}\lambda L) - \frac{\partial}{\partial x}\left(2\mathrm{i}\lambda^2 L + \lambda L \frac{\partial L}{\partial x} \right) + \left[\mathrm{i}\lambda L, 2\mathrm{i}\lambda^2 L + \lambda L \frac{\partial L}{\partial x} \right] = 0$$

Connection to the nonlinear Schrödinger equation (NLS)

- HM model has direct connection to the very well known model, focusing nonlinear Schrödinger equation. It is given by

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- The Lax pair for nonlinear Schrödinger equation is given by

$$\begin{cases}\dfrac{\partial\Psi}{\partial x}=(\mathrm{i}\lambda\sigma_3+A)\Psi\\ \dfrac{\partial\Psi}{\partial t}=\left(2\mathrm{i}\lambda^2\sigma_3+2\lambda A+\mathrm{i}|\psi|^2\sigma_3+B\right)\Psi\end{cases},$$

where $A=\mathrm{i}\begin{pmatrix}0 & \overline{\psi}\\ \psi & 0\end{pmatrix}$, $B=\begin{pmatrix}0 & \dfrac{\partial\overline{\psi}}{\partial x}\\ -\dfrac{\partial\psi}{\partial x} & 0\end{pmatrix}$, $\lambda\in\mathbb{C}$.

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- It was established in [ZT79] that after proper choice of g the functions Φ and Ψ are related by $\Phi = g\Psi$. Moreover,

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- As the result we observed that HM is equivalent to NLS and does not represent new model.

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- We introduce the anisotropy in spin interaction

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$$J = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix}, \quad 0 < J_1 < J_2 < J_3.$$

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- Landau-Lifshitz equation is obtained in the continuous limit of properly modified lattice model

$$\frac{\partial S}{\partial t} = S \times \frac{\partial^2 S}{\partial x^2} + S \times JS, \quad \sum_{j=1}^{3} S_j^2 = 1$$

$$S \to (0,0,1), \quad x \to \pm\infty$$

Lax pair

- The Lax pair is given by

$$\begin{cases} \dfrac{\partial \Psi}{\partial x} = U\Psi \\ \dfrac{\partial \Psi}{\partial t} = V\Psi \end{cases}, \quad \text{where}$$

$$U = -\mathrm{i}\sum_{j=1}^{3} \sigma_j S_j w_j, \quad V = 2\mathrm{i}\sum_{\substack{j,m,n=1\\ j\neq m\neq n}}^{3} \sigma_j S_j w_m w_n + \mathrm{i}\sum_{j=1}^{3} \sigma_j P_j w_j,$$

$$P_j = \left(\frac{\partial S}{\partial x} \times S\right)_j$$

$$w_1 = \rho\,\frac{1}{\mathrm{sn}(\lambda,k)}, \quad w_2 = \rho\,\frac{\mathrm{dn}(\lambda,k)}{\mathrm{sn}(\lambda,k)}, \quad w_3 = \rho\,\frac{\mathrm{cn}(\lambda,k)}{\mathrm{sn}(\lambda,k)}.$$

$$\rho = \frac{\sqrt{J_3 - J_1}}{2}, \quad k = \sqrt{\frac{J_2 - J_1}{J_3 - J_1}}$$ – elliptic modulus.

Features

- The key difference from the Heisenberg magnet is that the spectral parameter now belongs to the torus.

$$\lambda \in \mathbb{T}^2 = \{\lambda : |\mathrm{Re}(\lambda)| \leq 2K, |\mathrm{Im}(\lambda)| \leq 2K'\}$$

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- The Lax pair for the Landau-Lifshitz equation was found in [Skl79].
- In the next slides we will go over the inverse scattering method in more details.

Jost solution

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where $\mathcal{S}(\mathbb{R})$ is the Schwartz class.
- That implies that all solutions of equation

$$\frac{\partial \Psi}{\partial x}(\lambda, x) = U(\lambda, x)\Psi(\lambda, x), \quad U(\lambda, x) = -\mathrm{i}\sum_{j=1}^{3} \sigma_j S_j(x) w_j(\lambda)$$

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behave like planar waves for $x \to \pm\infty$ up to constant factor.
- We fix two solutions called Jost solutions

$$F_{\pm}(\lambda, x) \sim \mathrm{e}^{-\mathrm{i}w_3(\lambda)x\sigma_3}, \quad , x \to \pm\infty$$

Symmetries

- The following shift properties hold

$$w_1(\lambda+2K)=-w_1(\lambda),\quad w_1(\lambda+2\mathrm{i}K')=w_1(\lambda),$$
$$w_2(\lambda+2K)=-w_2(\lambda),\quad w_2(\lambda+2\mathrm{i}K')=-w_2(\lambda),$$
$$w_3(\lambda+2K)=w_3(\lambda),\quad w_3(\lambda+2\mathrm{i}K')=-w_3(\lambda),$$

- They imply the symmetries of coefficient matrix

$$\sigma_3 U(\lambda+2K,x)\sigma_3=U(\lambda,x),\quad \sigma_1 U(\lambda+2\mathrm{i}K',x)\sigma_1=U(\lambda,x).$$

- and the symmetries of Jost solutions

$$\sigma_3 F_\pm(\lambda+2K,x)\sigma_3=F_\pm(\lambda,x),\quad \sigma_1 F_\pm(\lambda+2\mathrm{i}K',x)\sigma_1=F_\mp(\lambda,x).$$

Scattering data

- By taking the ratio of Jost solutions we define the scattering matrix

$$S(\lambda) = (F_{-}(\lambda, x))^{-1} F_{+}(\lambda, x) = \begin{pmatrix} \mathrm{a}(\lambda) & -\overline{\mathrm{b}(\bar{\lambda})} \\ \mathrm{b}(\lambda) & \overline{\mathrm{a}(\bar{\lambda})} \end{pmatrix}.$$

- This formula suggests that the properties of scattering matrix are derived from the properties of Jost solutions.
- To construct the Jost solutions we would need to make the following transformation

$$\Upsilon_{\pm}(\lambda, x) = F_{\pm}(\lambda, x) \mathrm{e}^{\mathrm{i} x w_3(\lambda) \sigma_3} = \left(\upsilon_{\pm}^{(1)}(\lambda, x), \upsilon_{\pm}^{(2)}(\lambda, x) \right).$$

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- The result solves integral equation

$$\frac{\partial \Upsilon_\pm}{\partial x}(\lambda, x) = U(\lambda, x)\Upsilon_\pm(\lambda, x) + \mathrm{i}w_3(\lambda)\Upsilon_\pm(\lambda, x)\sigma_3,$$

Construction of Jost solutions

Lemma 1 ([Rod84])

The solutions $\widehat{\upsilon}_{\pm}^{(j)}(\lambda, x)$ *of the integral equations*

$$\widehat{\upsilon}_{\pm}^{(1)}(\lambda, x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{\pm\infty}^{x} \mathrm{e}^{\mathrm{i}(\mathbb{1}-\sigma_3)w_3(\lambda)(x-\tau)} \left(U(\lambda,\tau) + \mathrm{i}w_3(\lambda)\sigma_3\right)$$

$$\times\, \widehat{\upsilon}_{\pm}^{(1)}(\lambda, \tau) d\tau,$$

$$\widehat{\upsilon}_{\pm}^{(2)}(\lambda, x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{\pm\infty}^{x} \mathrm{e}^{-\mathrm{i}(\mathbb{1}+\sigma_3)w_3(\lambda)(x-\tau)} \left(U(\lambda,\tau) + \mathrm{i}w_3(\lambda)\sigma_3\right)$$

$$\times\, \widehat{\upsilon}_{\pm}^{(2)}(\lambda, \tau) d\tau,$$

coincide with functions $\upsilon_{\pm}^{(j)}(\lambda, x)$ *introduced earlier.*

Analytical properties of Jost solutions

- Let us denote

$$\Omega_+ = \left\{\lambda : 0 \leq \mathrm{Im}(\lambda) \leq 2K'; |\mathrm{Re}\lambda| \leq 2K\right\},$$
$$\Omega_- = \left\{\lambda : -2K' \leq \mathrm{Im}(\lambda) \leq 0; |\mathrm{Re}\lambda| \leq 2K\right\},$$
$$\Gamma_1 = \{\lambda \in \mathbb{T}^2 : \mathrm{Im}(\lambda) = 0\}$$
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Lemma 2 ([Rod84])

The functions $\upsilon_\pm^{(1)}(\lambda, x)$, $\upsilon_\mp^{(2)}(\lambda, x)$ are analytic in the domains $\Omega_\pm$ and bounded in the domains $\overline{\Omega}_\pm$ respectively. In addition $\upsilon_\pm^{(1)}(\lambda, x)$, $\upsilon_\pm^{(2)}(\lambda, x) \in C^\infty(\Gamma_1)$.

Convenient formulas for scattering data

Lemma 3

The functions a(λ), b(λ) *admit two alternative expressions.*

1. *They can be written as the integrals*

$$\begin{pmatrix} \mathrm{a}(\lambda) \\ \mathrm{b}(\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(\mathbb{1}-\sigma_3)w_3(\lambda)\tau} \left(U(\lambda,\tau) + \mathrm{i}w_3(\lambda)\sigma_3\right)$$

$$\times\, \upsilon_{-}^{(1)}(\lambda,\tau)d\tau.$$

2. *They can also be expressed as the following determinants*

$$\mathrm{a}(\lambda) = \det(\upsilon_{+}^{(1)}(\lambda,x), \upsilon_{-}^{(2)}(\lambda,x)),$$
$$\mathrm{b}(\lambda) = \mathrm{e}^{2\mathrm{i}w_3(\lambda)x} \det(\upsilon_{-}^{(1)}(\lambda,x), \upsilon_{+}^{(1)}(\lambda,x)).$$

Reflection coefficient

- We introduce the *reflection* coefficient

$$r(\lambda) = \frac{b(\lambda)}{a(\lambda)}.$$

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- Given the reflection coefficient, one can recover both $b(\lambda)$ and $a(\lambda)$ using formula
$$a(\lambda) = \exp\left\{-\frac{1}{2\pi i}\int_{-2K}^{0} \log\left(1+|r(\eta)|^2\right)\frac{w_3(\eta-\lambda)}{\rho}d\eta\right\}, \quad \lambda \in \Omega_+.$$

- Function $\frac{w_3(\eta-\lambda)}{\rho}$ plays role of Cauchy kernel. It has simple pole with residue 1 at zero.

Reflection coefficient properties

Lemma 4

Under the soliton free assumption, $\mathrm{a}(\lambda) \neq 0$ *the reflection coefficient* $\mathrm{r}(\lambda)$ *corresponding to the initial data from the Schwartz class satisfies*

1. $\mathrm{r}(\lambda) \in C^\infty(\Gamma_1 \cup \Gamma_2)$
2. $\mathrm{r}(\lambda + 2K) = -\mathrm{r}(\lambda)$
3. $\mathrm{r}(\lambda + 2\mathrm{i}K') = -\overline{\mathrm{r}(\bar{\lambda})}$
4. $\mathrm{r}(0) = 0$
5. $\frac{d^n \mathrm{r}(\lambda)}{d\lambda^n} = \mathcal{O}(\lambda^m), \quad \lambda \to 0, \quad \forall n, m \in \mathbb{N}.$

Riemann-Hilbert problem

- Introduce the following functions

$$Y_+(\lambda,x)=\left(\frac{\upsilon_+^{(1)}(\lambda,x)}{\mathrm{a}(\lambda)},\upsilon_-^{(2)}(\lambda,x)\right),$$

$$Y_-(\lambda,x)=\left(\upsilon_-^{(1)}(\lambda,x),\frac{\upsilon_+^{(2)}(\lambda,x)}{\overline{\mathrm{a}(\bar{\lambda})}}\right),$$

Riemann-Hilbert problem

Riemann-Hilbert problem 1.

1. The function $Y(\lambda, x)$ is bounded, doubly periodic, and piecewise analytic for $\lambda \in \mathbb{T}^2/(\Gamma_1 \cup \Gamma_2)$. Orientation of the contours Γ_1, Γ_2 is as specified below.

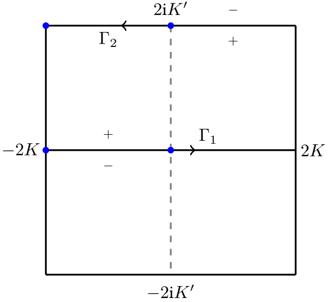

Figure 1: Contours Γ_1 and Γ_2.

Riemann-Hilbert problem

2. For $\lambda \in \Gamma_1, \Gamma_2$, the following jump condition holds

$$Y_+(\lambda, x) = Y_-(\lambda, x) G(\lambda, x),$$
$$G(\lambda, x) = \begin{pmatrix} 1 + |r(\lambda)|^2 & \overline{r(\lambda)} e^{-2ixw_3(\lambda)} \\ r(\lambda) e^{2ixw_3(\lambda)} & 1 \end{pmatrix}.$$

3. The function $Y(\lambda, x)$ satisfies the following symmetry conditions

$$\sigma_3 Y(\lambda + 2K, x)\sigma_3 = Y(\lambda, x), \quad \sigma_1 Y(\lambda + 2iK', x)\sigma_1 = Y(\lambda, x).$$

4. Function $Y(\lambda, x)$ satisfies normalization condition

$$\det(Y(\lambda, x)) = 1$$

Uniqueness

Lemma 1

Solution $Y(\lambda, x)$ of the RHP 1 is unique up to a sign.

Theorem 2

Let $Y(\lambda, x)$ be the solution of the Riemann-Hilbert problem 1 with $\mathrm{r}(\lambda)$ *satisfying properties* (1)-(5). *Then the function $S(x)$ constructed from it by formula*

$$Y(0,x)\sigma_3\left(Y(0,x)\right)^{-1} = \sum_{j=1}^{3} S_j(x)\sigma_j,$$

belongs to the Schwartz class: $S_1(x), S_2(x), S_3(x) - 1 \in \mathcal{S}(\mathbb{R})$, and it defines the initial data for the LL equation whose reflection coefficient is given by $\mathrm{r}(\lambda)$.

Time evolution

Lemma 3 ([Skl79])

Assume $t > 0$ is fixed and

$$S_1(x,t), S_2(x,t), S_3(x,t) - 1 \in \mathcal{S}(\mathbb{R})$$

are solutions of LL equation. Denote by $F_\pm(\lambda, x, t)$ the Jost solutions corresponding to $S(x,t)$ with $t > 0$. Then the matrix-valued functions

$$J_\pm(\lambda, x, t) = F_\pm(\lambda, x, t)\mathrm{e}^{2\mathrm{i}tw_1(\lambda)w_2(\lambda)\sigma_3}$$

solves equations of Lax pair of LL equation. Moreover the scattering data depends on time as

$$\mathrm{a}(\lambda, t) = \mathrm{a}(\lambda), \quad \mathrm{b}(\lambda, t) = \mathrm{b}(\lambda)\mathrm{e}^{-4\mathrm{i}tw_1(\lambda)w_2(\lambda)}.$$

Riemann-Hilbert problem corresponding to LL dynamics

Riemann-Hilbert problem 2.

1. The function $Y(\lambda, x, t)$ is bounded and piecewise analytic for $\lambda \in \mathbb{T}^2/(\Gamma_1 \cup \Gamma_2)$.
2. For $\lambda \in \Gamma_1, \Gamma_2$, the following jump condition holds

$$Y_+(\lambda, x, t) = Y_-(\lambda, x, t) G(\lambda, x, t)$$

$$G(\lambda, x, t) = \begin{pmatrix} 1 + |\mathrm{r}(\lambda)|^2 & \overline{\mathrm{r}(\lambda)} \mathrm{e}^{-2\mathrm{i}tp(\lambda,\varkappa)} \\ \mathrm{r}(\lambda) \mathrm{e}^{2\mathrm{i}tp(\lambda,\varkappa)} & 1 \end{pmatrix}.$$

where

$$p(\lambda, \varkappa) = \varkappa w_3(\lambda) - 2w_1(\lambda) w_2(\lambda), \quad \varkappa = \frac{x}{t}.$$

Riemann-Hilbert problem corresponding to LL dynamics

3. The function $Y(\lambda, x, t)$ satisfies the following symmetry conditions

$$\sigma_3 Y(\lambda + 2K, x, t)\sigma_3 = Y(\lambda, x, t), \quad \sigma_1 Y(\lambda + 2\mathrm{i}K', x, t)\sigma_1 = Y(\lambda, x,$$

4. Function $Y(\lambda, x, t)$ satisfies normalization condition

$$\det(Y(\lambda, x, t)) = 1.$$

Solution of LL from RHP

Lemma 4

Given $Y(\lambda, x, t)$ *is a solution of the Riemann-Hilbert problem 2 the function*

$$\Psi(\lambda, x, t) = Y(\lambda, x, t)\mathrm{e}^{-\mathrm{i}tp(\lambda,\varkappa)\sigma_3}$$

solves Lax pair of LL equation with $S_j(x, t)$ *given by*

$$Y(0, x, t)\sigma_3 \left(Y(0, x, t)\right)^{-1} = \sum_{j=1}^{3} S_j(x, t)\sigma_j. \qquad (1)$$

Main result

Theorem 5 ([?])

Let $Y(\lambda, x, t)$ *be the solution of the Riemann-Hilbert problem 2 with* $\mathrm{r}(\lambda)$ *satisfying the properties (1)-(5) and let the vector function* $S(x, t)$ *be determined by equation (1). Then,* $S(x, t)$ *solves the Cauchy problem for the LL equation characterized by the reflection coefficient* $\mathrm{r}(\lambda)$ *and for* $t \to \infty$, $\quad 0 < m \leq \frac{x}{t} \leq M$,

$$S_1(x,t) = \frac{1}{\rho}\left(\frac{2\nu}{t\varphi_0}\right)^{1/2} w_2(\lambda_0)\cos\theta(x,t) + \mathcal{O}(t^{-\frac{2}{3}}),$$

$$S_2(x,t) = \frac{1}{\rho}\left(\frac{2\nu}{t\varphi_0}\right)^{1/2} w_1(\lambda_0)\sin\theta(x,t) + \mathcal{O}(t^{-\frac{2}{3}}),$$

$$S_3(x,t) = 1 - \frac{1}{2}\left(S_1^2(x,t) + S_2^2(x,t)\right) + \mathcal{O}(t^{-\frac{7}{6}}),$$

Parameters

where

$$\theta(x,t) = 2tp(\lambda_0, \varkappa) + \nu \log t - \frac{\pi}{4} - \arg\Gamma(\mathrm{i}\nu) + \arg \mathrm{r}_0 - 2\mathrm{c}_0 + \nu \log\left(\frac{2\varphi_0}{\beta_0^2}\right),$$

$$p(\lambda, \varkappa) = \varkappa\, w_3(\lambda) - 2w_1(\lambda)w_2(\lambda), \quad \varkappa = \frac{x}{t}$$

and the value of the stationary point $\lambda_0 \in [-2K, 0]$ is determined by the equation $\partial_\lambda p(\lambda_0, \varkappa) = 0$. With such λ_0, the parameter $\varphi_0 = -\partial_\lambda^2 p(\lambda_0, \varkappa)$ is obtained from

$$\varphi_0 = \frac{1}{\rho^2}\left(8w_1(\lambda_0)w_2(\lambda_0)w_3^2(\lambda_0) + (w_1^2(\lambda_0) + w_2^2(\lambda_0))(2w_1(\lambda_0)w_2(\lambda_0) - \varkappa w_3(\lambda_0))\right),$$

Parameters

- The remaining terms are determined as follows:

$$\mathrm{r}_0 = \mathrm{r}(\lambda_0), \qquad \nu = \frac{1}{2\pi}\log(1+|\mathrm{r}_0|^2).$$

$$\beta(\lambda) = \frac{\sigma(\lambda)\sigma(\lambda-2K)}{\sigma(\lambda+2\mathrm{i}K')\sigma(\lambda-2\mathrm{i}K'-2K)},$$

$$\beta_0 := \frac{\sigma(-2K)}{\sigma(2\mathrm{i}K')\sigma(-2\mathrm{i}K'-2K)},$$

$$\mathrm{c}_0 = \frac{1}{2\pi}\int_{\lambda_0}^{0} d\left(\log\left(1+|\mathrm{r}(\eta)|^2\right)\right)\log\beta(\eta-\lambda_0),$$

  where $\sigma(\lambda)$ denotes the Weierstrass sigma function.
- This result was obtained in [BI88] with much lower level of rigor.

Future work

- Soliton gas analysis for the LL equation.
- Rogue waves of infinite order.
- Approximation theory on the torus with the goal to optimize numerical computation of singular integrals of type

$$\int_0^{2K} f(\mu)C(\mu,\lambda-\mathrm{i}0)d\mu$$

where

$$C(\mu,\lambda) := \zeta(\mu-\lambda)-\zeta(\mu-\mathrm{i}K')+\zeta(\lambda-K-\mathrm{i}K')+\zeta(K),$$

and $\zeta(.)$ is the Weierstrass ζ-function.

[BI88] R. F. Bikbaev and A. R. Its.
Asymptotic behavior of the solution to the Cauchy problem as $t \to \infty$ for the Landau-Lifshits equation.
Teoret. Mat. Fiz., 76(1):3–17, 1988.

[DZ93] P. Deift and X. Zhou.
A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation.
Ann. of Math. (2), 137(2):295–368, 1993.

[FT07] Ludwig D. Faddeev and Leon A. Takhtajan.
Hamiltonian methods in the theory of solitons.
Classics in Mathematics. Springer, Berlin, english edition, 2007.
Translated from the 1986 Russian original by Alexey G. Reyman.

[Rod84] Yu. L. Rodin.
The Riemann boundary problem on Riemann surfaces and the inverse scattering problem for the Landau-Lifschitz equation.
Phys. D, 11(1-2):90–108, 1984.

[Skl79] E K Sklyanin.
On complete integrability of the Landau–Lifshitz equation.
Technical report, 1979.

[ZT79] V. E. Zakharov and L. A. Takhtajan.
Equivalence of a nonlinear Schrödinger equation and a Heisenberg ferromagnet equation.
Teoret. Mat. Fiz., 38(1):26–35, 1979.

Thank You