On some Hamiltonian properties of isomonodromic tau functions.

Andrei Prokhorov

Joint work with Alexander Its

In the memory of L. D. Faddeev

Indiana University-Purdue University Indianapolis

Saint-Petersburg State University

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Painlevé equations

- The Painlevé equations was written first time by Painlevé (1900) and Gambier (1910) during their studies of ODEs, whose solution's branching points are independent on initial conditions.
- For example the equation Painlevé-IV is given by

$$q_{tt} = \frac{q_t^2}{2q} + \frac{3}{2}q^3 + 4tq^2 + 2(t^2 - 2\theta_\infty + 1)q - \frac{8\theta_0^2}{q}. \tag{1}$$

- This equation is equivalent to the Hamiltonian system with Hamiltonian

$$H = 2p^2q + p(q^2 + 2qt + 4\theta_0) + q(\theta_0 + \theta_\infty).$$

Normal form of the Hamiltonian.

- Takasaki (2000) noticed that if we make change of variable $u(t) = 2e^{i\frac{\pi}{4}}\sqrt{q(it)}$ we get the equation

$$u_{tt} = \frac{3u^5}{64} + \frac{t}{2}u^3 + (t^2 - 2\theta_\infty + 1)u - \frac{4\theta_0^2}{u^3}.$$

- This equation is equivalent to the Hamiltonian system with the Hamiltonian in the normal form

$$H = \frac{p^2}{2} + V(x,t),$$

with potential

$$V(x,t) = -\frac{x^6}{128} - \frac{t}{8}x^4 - (t^2 - 2\theta_\infty + 1)\frac{x^2}{2} - \frac{16\theta_0^2}{x^2}$$

Connection problem

- The Riemann-Hilbert approach provides us with asymptotic for solutions of Painlevé equations as t approaches infinity. The asymptotic is parametrised by monodromy data.
- The natural question is to study the asymptotic of tau function when t_1 and t_2 approach infinity in different directions in complex plane.
- The connection problem consists in determining such asymptotics.
- Using the asymptotic for solutions of Painlevé equations we can get the asymptotic for tau function up to the term independent of t_1, t_2. To find this term is more complicated problem.

Different results

- Iorgov, Lisovyy, Tykhyy(2013), Its, Lisovyy, Tykhyy(2014), Lisovyy, Nagoya, Roussillon(2018) got the conjectural results for PVI, PIII, PV using the quasiperiodicity of the connection constant and its interpretation as generating function for canonical transformation.
- Its, P.(2016), Lisovyy, Roussillon (2017), Its, Lisovyy, P.(2018) got the results for PIII, PI, PVI, PII using the extension of JMU form suggested by Bertola based on works by Malgrange.
- Bothner, Its, P.(2017), Bothner (2018) got the results for PII, PIII, PV using interpretation of extension of JMU in terms of an action.
- The main result of authors is relation with action for all Painlevé equations and Schlesinger equation. In these slides we consider Painlevé-IV case.

Lax pair

- The Lax pair for Painlevé-IV case is given by (see Jimbo Miwa, 1981)

$$\frac{d\Psi}{dz} = A(z)\Psi(z), \qquad \frac{d\Psi}{dt} = B(z)\Psi(z)$$

$$A(z) = A_1 z + A_0 + \frac{A_{-1}}{z}, \quad B(z) = B_1 z + B_0$$

$$A_0 = \begin{pmatrix} t & k \\ \frac{2(r-\theta_0-\theta_\infty)}{k} & -t \end{pmatrix}, \quad A_{-1} = \frac{1}{z}\begin{pmatrix} -r+\theta_0 & -\frac{kq}{2} \\ \frac{2r(r-2\theta_0)}{kq} & r-\theta_0 \end{pmatrix},$$

$$A_1 = B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & k \\ \frac{2(r-\theta_0-\theta_\infty)}{k} & 0 \end{pmatrix}.$$

The compatibility condition

- The compatibility condition for the Lax pair has form

$$\frac{dA}{dt} - \frac{dB}{dz} + [A, B] = 0.$$

- It is equivalent to the system

$$\begin{cases} \dfrac{dq}{dt} = -4r + q^2 + 2tq + 4\theta_0, \\ \dfrac{dr}{dt} = -\dfrac{2}{q} r^2 + \left(-q + \dfrac{4\theta_0}{q}\right) r + (\theta_0 + \theta_\infty) q, \\ \dfrac{dk}{dt} = -k(q + 2t). \end{cases}$$

- The function $q(t)$ satisfies Painlevé-IV equation (1).

Local behavior of Ψ-function at infinity

- The first equation of the Lax pair has irregular singularity of Poincaré rank 2 at infinity.
- We have the following formal solution at infinity

$$\Psi_\infty(z) = G_\infty(z) e^{\Theta_\infty(z)}, \quad \Theta_\infty(z) = \sigma_3 \left(\frac{z^2}{2} + tz - \theta_\infty \ln z \right),$$

$$G_\infty(z) = \left(I + \frac{g_1}{z} + \frac{g_2}{z^2} + O\left(\frac{1}{z^3}\right) \right), \quad z \to \infty$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Local behavior of Ψ-function at zero

- The first equation of the Lax pair has regular singularity at zero.
- We have the following solution at zero

$$\Psi_0(z) = G_0(z) z^{\theta_0 \sigma_3}, \quad G_0(z) = P_0 \left(I + O\left(z \right) \right), \quad z \to 0,$$

$$P_0 = \frac{1}{2\sqrt{kq\theta_0}} \begin{pmatrix} -kq & -kq \\ 2r & 2r - 4\theta_0 \end{pmatrix} a^{-\frac{\sigma_3}{2}}.$$

- To satisfy the second equation of the Lax pair we need to have

$$\frac{da}{dt} = \frac{4\theta_0}{q} a.$$

The Jimbo-Miwa-Ueno form

- The Jimbo-Miwa-Ueno form is given by

$$\omega_{\mathrm{JMU}} = -\operatorname{res}_{z=\infty} \operatorname{Tr}\left(\left(G_\infty(z)\right)^{-1}\frac{dG_\infty(z)}{dz}\frac{d\Theta_\infty(z)}{dt}\right)dt$$
$$= -\operatorname{Tr}\left(g_1\sigma_3\right)dt$$
$$= \left[\frac{2}{q}r^2 - \left(q + 2t + \frac{4\theta_0}{q}\right)r + (\theta_0+\theta_\infty)(r+2t)\right]dt$$
$$= \left(\frac{q_t^2}{8q} - \frac{q^3}{8} - \frac{q^2 t}{2} - \frac{qt^2}{2} - \frac{2\theta_0^2}{q} + \theta_\infty q + 2\theta_\infty t\right)dt$$

- In general

$$\omega_{\mathrm{JMU}} = -\sum_{k=1}^{L}\sum_{a_\nu}\operatorname{res}_{z=a_\nu}\operatorname{Tr}\left(\left(G_\nu(z)\right)^{-1}\frac{dG_\nu(z)}{dz}\frac{d\Theta_\nu(z)}{dt_k}\right)dt_k$$

The isomonodromic tau function

- The isomonodromic tau function is given by

$$\ln \tau^{JMU}(t_1, t_2) = \int_{t_1}^{t_2} \omega_{JMU}.$$

- We have the relation

$$\ln \tau^{JMU}(t_1, t_2) = \ln \tau^{O}(t_1, t_2) + \int_{t_1}^{t_2} q dt + (\theta_0 + \theta_\infty)(t_2^2 - t_1^2).$$

Hamiltonian structure

- We expect

$$\omega_{JMU} \simeq H dt.$$

- Unfortunately if we choose the Hamiltonian in such way, r and q are not Darboux coordinates for Hamiltonian dynamics.

$$\omega_{JMU} = \left[\frac{2}{q} r^2 - \left(q + 2t + \frac{4\theta_0}{q}\right) r + (\theta_0 + \theta_\infty)(r + 2t)\right] dt.$$

$$\begin{cases} \dfrac{dq}{dt} = -4r + q^2 + 2tq + 4\theta_0, \\ \dfrac{dr}{dt} = -\dfrac{2}{q} r^2 + \left(-q + \dfrac{4\theta_0}{q}\right) r + (\theta_0 + \theta_\infty) q. \end{cases}$$

Hamiltonian structure

- Hamiltonian structure for Painlevé equations was introduced by Okamoto (1980). It was interpreted in terms of moment map and Hamiltonian reduction in the dual loop algebra $\widetilde{sl_2(\mathbb{R})}^*$ in the work by Harnad and Routhier(1995).
- We want to study the Hamiltonian structure using the extension of Jimbo-Miwa-Ueno form, following works of Bertola (2010), Malgrange(1983), Its, Lisovyy, P.(2018).

Symplectic form

- Consider the configuration space for Painlevé-IV Lax pair consisting of coordinates

$$\{q, r, k, a, \theta_0, \theta_\infty\}.$$

- We denote by δ the differential in this space.
- Following the work of Its, Lisovyy, P.(2018) consider the form

$$\omega_0 = \mathrm{res}_{z=\infty} \mathrm{Tr} \left(A(z) \, \delta G_\infty(z) \, G_\infty(z)^{-1} \right)$$

$$+ \mathrm{res}_{z=0} \mathrm{Tr} \left(A(z) \, \delta G_0(z) \, G_0(z)^{-1} \right) = \mathrm{Tr}(A_{-1} \delta G_0 G_0^{-1} - A_1 \delta g_2$$

$$+ A_1 \delta g_1 g_1 - A_0 \delta g_1)$$

Symplectic form

- In general this form is given by

$$\omega_0 = \sum_{a_\nu} \mathrm{res}_{z=a_\nu} \operatorname{Tr}\left(A(z)\,\delta G_\nu(z)\, G_\nu(z)^{-1}\right).$$

- In all examples considered the symplectic form for Hamiltonian dynamics was given by

$$\Omega_0 = \delta\omega_0.$$

- In case of Painlevé-IV we have

$$\Omega_0 = -\frac{1}{q}\delta r \wedge \delta q + \frac{1}{k}\delta k \wedge \delta\theta_\infty + \frac{1}{a}\delta a \wedge \delta\theta_0 - \frac{1}{q}\delta q \wedge \delta\theta_0.$$

Darboux coordinates

- We can choose Darboux coordinates as

$$p_1 = -\frac{r}{q}, \quad q_1 = q,$$

$$p_2 = \ln k = -\int_{c_1}^{t} (q + 2t) dt, \quad q_2 = \theta_\infty$$

$$p_3 = \ln a - \ln q = \int_{c_2}^{t} \frac{4\theta_0}{q} dt - \ln q, \quad q_3 = \theta_0.$$

Hamiltonian

- Jimbo-Miwa-Ueno form in these coordinates take form

$$\omega_{JMU} = \left(2p_1^2 q_1 + p_1(q_1^2 + 2q_1 t + 4q_3) + (q_1 + 2t)(q_3 + q_2)\right) dt$$

- The deformation equations take form

$$\begin{cases} \dfrac{dq_1}{dt} = 4p_1 q_1 + q_1^2 + 2q_1 t + 4q_3, & \dfrac{dp_3}{dt} = -4p_1 - q_1 - 2t, \\ \dfrac{dp_1}{dt} = -2p_1^2 - 2p_1 q_1 - 2p_1 t - q_3 - q_2, & \dfrac{dp_2}{dt} = -(q_1 + 2t). \end{cases}$$

- These equations become Hamiltonian system with Hamiltonian given by the equation

$$\omega_{JMU} = H dt$$

Counterexample

- We can choose Darboux coordinates in different way

$$\tilde{p}_1 = -\frac{r}{q} + f(t), \quad q_1 = q,$$

$$p_2 = \ln k = -\int_{c_1}^{t} (q + 2t)dt, \quad q_2 = \theta_\infty$$

$$p_3 = \ln a - \ln q = \int_{c_2}^{t} \frac{4\theta_0}{q} dt - \ln q, \quad q_3 = \theta_0.$$

Counterexample

- Jimbo-Miwa-Ueno form in these coordinates take form

$$\omega_{JMU} = \left(2(\tilde{p}_1 - f)^2 q_1 + (\tilde{p}_1 - f)(q_1^2 + 2q_1 t + 4q_3) + (q_1 + 2t)(q_3 + q_2)\right.$$

- The deformation equations take form

$$\begin{cases} \dfrac{dq_1}{dt} = 4\tilde{p}_1 q_1 - 4fq_1 + q_1^2 + 2q_1 t + 4q_3, \ \dfrac{dp_3}{dt} = -4\tilde{p}_1 + 4f - q_1 - 2t, \\ \dfrac{dp_1}{dt} = -2(\tilde{p}_1 - f)^2 - 2(\tilde{p}_1 - f)(q_1 + t) - q_3 - q_2 + f', \\ \dfrac{dp_2}{dt} = -(q_1 + 2t). \end{cases}$$

- These equations become Hamiltonian system with Hamiltonian given by the equation

$$\omega_{JMU} = (\tilde{H} - q_1 f')dt$$

Hamiltonian

- We can ask, what Hamiltonian induce isomonodromic deformation with respect to described symplectic structure.
- Consider the form in the configuration space

$$\alpha = \sum_{a_\nu} \mathrm{res}_{z=a_\nu} \operatorname{Tr}\left(\frac{\partial A(z)}{\partial t}\delta G_\nu(z)\, G_\nu(z)^{-1}\right)$$

$$-\sum_{a_\nu} \mathrm{res}_{z=a_\nu} \operatorname{Tr}\left(\frac{d\left(\delta\Theta_\nu(z)\right)}{dz} G_\nu(z)^{-1}\frac{\partial G_\nu(z)}{\partial t}\right)$$

Conjecture

The form α is exact and the Hamiltonian is given by

$$\alpha = \delta H.$$

Extension of Jimbo-Miwa-Ueno form

- We consider the extended configuration space. For Painlevé-IV it has coordinates

$$\{t, q_1, p_1, q_2, p_2, q_3, p_3\}$$

- We denote by "d" the differential in this space.
- Following Its, Lisovyy, P.(2018) we consider the form

$$\omega = \mathrm{res}_{z=\infty} \mathrm{Tr} \left(A(z)\, dG_\infty(z)\, G_\infty(z)^{-1} \right)$$

$$+ \mathrm{res}_{z=0} \mathrm{Tr} \left(A(z)\, dG_0(z)\, G_0(z)^{-1} \right) = \mathrm{Tr}(A_{-1} dG_0 G_0^{-1} - A_1 dg_2$$

$$+ A_1 dg_1 g_1 - A_0 dg_1).$$

Extension of Jimbo-Miwa-Ueno form

- In general this form is given by

$$\omega = \sum_{a_\nu} \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(A(z)\, dG_\nu(z)\, G_\nu(z)^{-1}\right).$$

- Using the first choice of Darboux coordinates and Hamiltonian we can rewrite it for Painlevé-IV case as

$$\omega = p_1 dq_1 + p_2 dq_2 + p_3 dq_3 - Hdt$$
$$+ d\left(\frac{Ht}{2} - \frac{p_1 q_1}{2} - p_2 q_2 - p_3 q_3 + \frac{q_3^2}{2} - \frac{q_3}{2} - \frac{q_2^2}{2} + \frac{q_2}{2}\right) \qquad (2)$$

Relation to action integral

- Let's return to the notations in terms of Painlevé-IV equation

$$q_1 = q, \quad p_1 = \frac{1}{4q}\left(q' - q^2 - 2qt - 4\theta_0\right),$$

$$q_2 = \theta_\infty, \quad p_2 = -\int\limits_{c_1}^{t} q dt + c_1^2 - t^2,$$

$$q_3 = \theta_0, \quad p_3 = \int\limits_{c_2}^{t} \frac{4\theta_0}{q} dt - \ln q,$$

$$H = 2p^2 q + p(q^2 + 2qt + 4\theta_0) + (q + 2t)(\theta_0 + \theta_\infty).$$

- Writing the "dt" part of the formula (2) we get the identity

$$H = pq' - H + \frac{1}{2}(Ht - pq)' - 4p\theta_0 - (q + 2t)(\theta_0 + \theta_\infty)$$

Relation to action integral

- We introduce the action integral

$$S(t_1, t_2) = \int_{t_1}^{t_2} (pq' - H)dt.$$

- We have the following formula as the result of the identity above

$$\ln \tau_{JMU}(t_1, t_2) = S(t_1, t_2) + \frac{1}{2}(Ht - pq)\Big|_{t_1}^{t_2}$$

$$- \int_{t_1}^{t_2} (4p\theta_0 + (q + 2t)(\theta_0 + \theta_\infty))dt.$$

Properties of action integral

- Assume the monodromy data is parametrized by coordinates $\{m_1, m_2, \theta_0, \theta_\infty\}$.
- The action integral is better then tau function, because

$$\frac{\partial S}{\partial m_1}(t_1, t_2) = \int_{t_1}^{t_2} \left(\frac{\partial p}{\partial m_1} q' + p \frac{\partial q'}{\partial m_1} - \frac{\partial H}{\partial p} \frac{\partial p}{\partial m_1} - \frac{\partial H}{\partial q} \frac{\partial q}{\partial m_1} \right) dt$$

$$= p \frac{\partial q}{\partial m_1} \bigg|_{t_1}^{t_2}.$$

Properties of action integral

- Similarly, following the idea of Bothner (2018), we have

$$\frac{\partial S}{\partial \theta_0}(t_1, t_2) = p\frac{\partial q}{\partial \theta_0}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} (4p + q + 2t)dt,$$

$$\frac{\partial S}{\partial \theta_\infty}(t_1, t_2) = p\frac{\partial q}{\partial \theta_\infty}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} (q + 2t)dt.$$

Relation to action integral

Main result

$$\ln \tau_{JMU}(t_1,t_2) = S(t_1,t_2) + \theta_0 \frac{\partial S}{\partial \theta_0}(t_1,t_2) + \theta_\infty \frac{\partial S}{\partial \theta_\infty}(t_1,t_2)$$

$$+ \frac{1}{2}(Ht - pq)\Big|_{t_1}^{t_2} - \theta_0 p \frac{\partial q}{\partial \theta_0}\Big|_{t_1}^{t_2} - \theta_\infty p \frac{\partial q}{\partial \theta_\infty}\Big|_{t_1}^{t_2}.$$

$$S(t_1,t_2) = \int_{(m_1^{(0)},m_2^{(0)})}^{(m_1,m_2)} p\frac{\partial q}{\partial m_1}\Big|_{t_1}^{t_2} dm_1 + p\frac{\partial q}{\partial m_2}\Big|_{t_1}^{t_2} dm_2.$$

- That formula is the good tool for computing connection constant up to numerical constant. Finding numerical constant is still complicated problem.

General case

- In the general case we have (see Its, Lisovyy, P.(2018)).

$$\ln \tau_{JMU}(t^{(\vec{1})}, t^{(\vec{2})})$$

$$= \int_{t^{(\vec{1})}}^{t^{(\vec{2})}} - \sum_{k=1}^{L} \sum_{a_\nu} \mathrm{res}_{z=a_\nu} \operatorname{Tr}\left(\left(G_\nu(z) \right)^{-1} \frac{dG_\nu(z)}{dz} \frac{d\Theta_\nu(z)}{dt_k} \right) dt_k$$

$$= \int_{\vec{m}}^{\vec{m}_0} \sum_{k=1}^{M} \sum_{a_\nu} \mathrm{res}_{z=a_\nu} \operatorname{Tr}\left(A(z) \frac{\partial G_\nu}{\partial m_k}(z)\, G_\nu(z)^{-1} \right) \Bigg|_{t^{(\vec{1})}}^{t^{(\vec{2})}} dm_k .$$