Asymptotics of tau-function for Painlevé equations

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Lax pair

Consider Flaschka-Newell Lax pair for Painlevé-II equation

$$\begin{cases} \dfrac{d\Psi}{dz} = A(z,t)\Psi(z,t) \\ \dfrac{d\Psi}{dt} = U(z,t)\Psi(z,t) \end{cases}$$

$$A(z) = -i4z^2\sigma_3 - 4qz\sigma_2 - qu_t\sigma_1 - it\sigma_3 - i2q^2\sigma_3,$$

$$U(z) = -iz\sigma_3 - q\sigma_2.$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Painlevé-II equation

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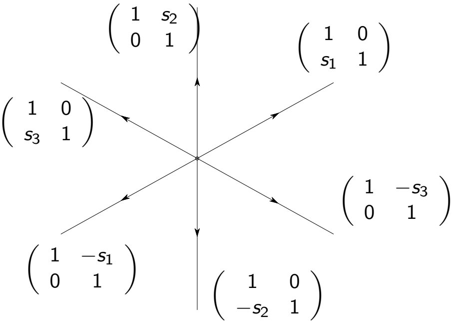
$$A(z) = -i4z^2 + -4qz\sigma_2 - 2q_t\sigma_1 - it\sigma_3 - i2q^2\sigma_3,$$

$$U(z) = -iz\sigma_3 - q\sigma_2.$$

$$\frac{dA}{dt} = \frac{dU}{dz} + [U, A] \Leftrightarrow q_{tt} = tq + 2q^3.$$

Riemann-Hilbert problem



Contour Γ.

$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0.$$

$$\Psi_+(z) = \Psi_-(z) S(z),$$

$$\Psi(z) = \left(I + \frac{m_1}{z} + O\left(\frac{1}{z^2}\right)\right) e^{-\left(\frac{4iz^3}{3} + izt\right)\sigma_3}, \quad z \to \infty,$$

$$q(t) = 2(m_1)_{1,2}.$$

Monodromy data

We make assumption on the monodromy data

$$\arg(1 - s_1 s_3) \in (-\pi, \pi), \qquad n \in \mathbb{Z},$$

$$\arg(i\sigma s_2) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \sigma = \operatorname{sgn} \Re(is_2) = \pm 1, \quad n \in \mathbb{Z}, .$$

Asymptotics of Painlevé function

Function $q(t)$ exhibits the following behaviour

$$q\left(t\right) = a_{+} e^{\frac{2}{3} i (-t)^{\frac{3}{2}}} (-t)^{\frac{3}{2}\mu - \frac{1}{4}} + a_{-} e^{-i \frac{2}{3} (-t)^{\frac{3}{2}}} (-t)^{-\frac{3}{2}\mu - \frac{1}{4}} + O\left(t^{3|\Re\mu| - 1}\right),$$

$$t \to -\infty,$$

$$q\left(t\right) = i\sigma\sqrt{\frac{t}{2}} + b_{+} e^{i \frac{2\sqrt{2}}{3} t^{\frac{3}{2}}} t^{-\frac{3}{2}\nu - \frac{1}{4}} + b_{-} e^{-i \frac{2\sqrt{2}}{3} t^{\frac{3}{2}}} t^{\frac{3}{2}\nu - \frac{1}{4}} + O\left(t^{3|\Re\nu| - 1}\right),$$

$$t \to +\infty,$$

A. Its, A. Kapaev (1987); P. Deift, X. Zhou (1995);
A. Kapaev (1996);

Connection formulae

$$\mu = -\frac{1}{2\pi i}\ln(1 - s_1 s_3),$$

$$a_+ = \frac{\sqrt{\pi}e^{-i\frac{\pi}{2}\mu}8^{\mu}e^{-i\frac{\pi}{4}}}{s_1\Gamma(\mu)}, \quad a_- = \frac{\sqrt{\pi}e^{-i\frac{\pi}{2}\mu}8^{-\mu}e^{i\frac{\pi}{4}}}{s_3\Gamma(-\mu)},$$

$$\nu = \frac{1}{\pi i}\ln(i\sigma s_2),$$

$$b_+ = \frac{\sqrt{\pi}e^{i\frac{\pi}{2}\nu}2^{-\frac{3}{4}-\frac{7}{2}\nu}e^{-i\frac{3\pi}{4}}i\sigma}{(1 + s_2 s_3)\Gamma(-\nu)}, \quad b_- = \frac{\sqrt{\pi}e^{i\frac{\pi}{2}\nu}2^{-\frac{3}{4}+\frac{7}{2}\nu}e^{i\frac{3\pi}{4}}i\sigma}{(1 + s_1 s_2)\Gamma(\nu)}.$$

Hamiltonian formulation

Introduce Hamiltonian

$$H = \frac{p^2}{4} - tq^2 - q^4.$$

$p = 2q_t$ plays role of the momentum, q plays role of the coordinate.

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p}, \\ \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q}, \end{cases} \Leftrightarrow q_{tt} = tq + 2q^3.$$

Introduce tau-function

$$\ln \tau(t) = \int_{t_0}^{t} H(y) dy$$

Asymptotics of tau-function

Plugging asymptotics of $q(t)$ in formula for $\tau(t)$ we get

$$\tau(t) \simeq \begin{cases} C_- e^{-\frac{4}{3} i\mu(-t)^{\frac{3}{2}}} (-t)^{-\frac{3}{2}\mu^2} & \text{as } t \to -\infty, \\ C_+ e^{\frac{t^3}{12} + \frac{2i\sqrt{2}}{3}\nu t^{\frac{3}{2}}} t^{-\frac{3}{4}\nu^2 - \frac{1}{8}} & \text{as } t \to +\infty. \end{cases}$$

Problem: Determine $\dfrac{C_+}{C_-}$.

Extension of differential form

We want to extend the form

$$Hdt = \left(q_t^2 - q^4 - tq^2\right) dt,$$

on variables $\{s_1, s_2\}$ in such a way, that it remains closed.

Extension of differential form

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B. Malgrange(1983) and M. Bertola(2010) provide the following construction

$$\begin{aligned} \omega_0 &= \left(q_t^2 - q^4 - tq^2\right) dt \\ &+\frac{2}{3}\left(2q_t q_{s_1} - 4q^3 tq_{s_1} - qq_{ts_1} + 2tq_t q_{ts_1} - 2qt^2 q_{s_1}\right) ds_1 \\ &+ \frac{2}{3}\left(2q_t q_{s_2} - 4q^3 tq_{s_2} - qq_{ts_2} + 2tq_t q_{ts_2} - 2qt^2 q_{s_2}\right) ds_2. \end{aligned}$$

Closedness

We have

$$d\omega_0 = (p_{s_2} q_{s_1} - p_{s_1} q_{s_2}) ds_2 \wedge ds_1 = dp \wedge dq,$$

This is symplectic form for Hamiltonian dynamics.
From Painlevé-II equation it follows that

$$\frac{d}{dt}(p_{s_2} q_{s_1} - p_{s_1} q_{s_2}) = 0,$$

and hence we can observe that

$$d\omega_0 = \lim_{t \to \pm\infty} d\omega = 4i da_- \wedge da_+ = 4i\sqrt{2} db_+ \wedge db_-.$$

It means that (a_+, a_-) and (b_+, b_-) play role of canonical coordinates at $\pm\infty$.

The result of previous observations is that the form $\omega = \omega_0 + 4ia_+ da_-$ is closed. We can define

$$\ln \tau(t, s_1, s_2) := \int \omega$$

This definition is unique up to closed differential in $\{s_1, s_2\}$. It does not affect on our calculation of $\frac{C_+}{C_-}$.

Asymptotics of form ω

$$\omega = d\left(-\frac{4i\mu}{3}(-t)^{\frac{3}{2}} - \frac{3}{2}\mu^2 \ln(-t) - \mu^2 - \mu\right)$$

$$+o(1), \qquad t \to -\infty,$$

$$\omega = d\left(\frac{t^3}{12} - \frac{6\nu^2+1}{8}\ln t + \frac{2i\sqrt{2}}{3}\nu t^{\frac{3}{2}} - \frac{\nu^2}{2} + \frac{\nu}{2}\right)$$

$$+4i\sqrt{2}b_+ db_- + 4ia_+ da_- + o(1), \qquad t \to +\infty.$$

Preliminary answer

$$\ln\left(\frac{C_+}{C_-}\right) = -\frac{\nu^2}{2} + \frac{\nu}{2} + \mu^2 + \mu + 4i\int a_+ da_- + \sqrt{2}b_+ db_- + c.$$

This is essentially generating function for canonical transformation $(a_+, a_-) \to (b_+, b_-)$. Introduce new variables

$$(1 + s_1 s_2)^{-1} = e^{i\pi\rho},$$

$$s_3^{-1} = e^{i\pi\eta}.$$

We can express the answer in terms of Barnes-G function.

Answer

Theorem

$$\frac{C_+}{C_-} = \mathrm{const} \cdot 2^{3\mu^2 - \frac{7}{4}\nu^2} (2\pi)^{-\mu - \frac{\nu}{2}} e^{\frac{\pi i}{4}(\eta^2 + 2\mu^2 + 2\eta\nu - 8\mu\eta - \sigma\eta - \sigma\nu + 4\sigma\mu)} \times$$

$$\times \frac{G(1+\eta-\frac{\sigma}{2})G(1-\nu)G^2(1+\frac{\sigma}{4}+\frac{\nu}{2}-\frac{\eta}{2})}{G(1+\frac{\sigma}{2}-\eta)G^2(1-\frac{\sigma}{4}-\frac{\nu}{2}+\frac{\eta}{2})G^2(1-\mu)}$$

Numeric constant

Consider Hastings-Mcleod solution $q_{\mathrm{HM}}(x)$, corresponding to the monodromy data

$$s_1 = -i, \quad s_2 = 0, \quad s_3 = i.$$

Then $\tilde{q}_{\mathrm{HM}}(t) = e^{\frac{2\pi i}{3}} q_{\mathrm{HM}}(e^{\frac{2\pi i}{3}} t)$ will correspond to monodromy data

$$s_3 = -i, \quad s_1 = 0, \quad s_2 = -i.$$

It satisfies our assumptions on the monodromy data. So to find numerical constant in $\dfrac{C_+}{C_-}$ we need to find the numerical constant for asymptotics of tau-function of $\tilde{q}_{\mathrm{HM}}(t)$, which is possible to do using Airy determinant representation.

Relation to classical action.

We can rewrite form ω_0 in the following compact form

$$\omega_0 = \left(q_t^2 - q^4 - tq^2\right) dt$$
$$+\frac{2}{3}\left(2q_t q_{s_1} - 4q^3 t q_{s_1} - q q_{t s_1} + 2t q_t q_{t s_1} - 2qt^2 q_{s_1}\right) ds_1$$
$$+\frac{2}{3}\left(2q_t q_{s_2} - 4q^3 t q_{s_2} - q q_{t s_2} + 2t q_t q_{t s_2} - 2qt^2 q_{s_2}\right) ds_2$$
$$= pdq - Hdt + d\left(\frac{2}{3}Ht - \frac{1}{3}pq\right).$$

This formula says that form Hdt coincide with the form of classical action up to a complete differential.

$$Hdt = pq_t dt - Hdt + \left(\frac{2}{3}Ht - \frac{1}{3}pq\right)_t dt.$$

Full asymptotic expansion

Consider

$$\tau_0(t) = \tau(t)^{\frac{1}{2}} e^{-\frac{t^3}{24}} e^{-\frac{1}{2}\int_{-\infty}^{t} q(y)dy}.$$

Then we conjecture that

$$\tau_0(t) = \sum_{k\in\mathbb{Z}} A(\mu+k,t)e^{i\pi\eta k}, \quad t\to-\infty$$

$$A(\mu,t) = G(1-\mu)2^{\frac{-3\mu^2}{2}}(2\pi)^{\frac{\mu}{2}} e^{-\frac{i\pi\mu^2}{4}} e^{i\pi\mu} e^{-\frac{t^3}{24}}(-t)^{-\frac{3\mu^2}{4}} e^{-\frac{2}{3}i\mu(-t)^{\frac{3}{2}}} B(\mu,t),$$

where $B(\mu,t)$ admits the asymptotic expansion

$$B(\mu,t) \sim 1 + \sum_{k=1}^{\infty} B_k(\mu)(-t)^{-\frac{3k}{2}}$$

Full asymptotic expansion

$$\tau_0(t) = \chi \sum_{k\in\mathbb{Z}} C(\nu - 2k, t) e^{i\pi\rho k}, \quad t \to +\infty$$

$$C(\nu,t) = G\left(\frac{\nu}{2} + 1 - \frac{\sigma}{2}\right) G\left(\frac{\nu}{2} + 1\right) 2^{-\frac{5}{8}\nu^2 + \frac{5\nu}{8}\sigma} (2\pi)^{-\frac{\nu}{2}} e^{-\frac{i\pi\nu^2}{8}} e^{\frac{i\pi\nu\sigma}{8}} \times$$

$$\times e^{\left(\frac{\sqrt{2}}{3} i\nu - \sigma \frac{i\sqrt{2}}{6}\right) t^{\frac{3}{2}}} t^{-\frac{3\nu^2}{8} + \sigma\frac{3\nu}{8} - \frac{1}{16}} D(\nu, t),$$

where $D(\nu, t)$ admits the asymptotic expansion,

$$D(\nu, t) \sim 1 + \sum_{k=1}^{\infty} D_k(\mu) t^{-\frac{3k}{2}}.$$

Quasiperiodicity

Taking into account the work of Baik,Buckingham,DiFranco,Its(2009) we get

$$\chi = \mathrm{const} \cdot 2^{-\frac{\nu^2}{4}} (2\pi)^{\frac{\nu}{4}+\frac{\sigma}{4}} e^{\frac{\pi i}{8}(\eta^2+\nu^2+2\eta\nu-8\mu\eta-3\sigma\eta-3\sigma\nu+8\mu(\sigma+1))}$$

$$\left(\frac{G(1+\frac{\sigma}{4}+\frac{\nu}{2}-\frac{\eta}{2})}{G(1-\frac{\sigma}{4}-\frac{\nu}{2}+\frac{\eta}{2})G\left(\frac{\nu}{2}+1-\frac{\sigma}{2}\right)G\left(\frac{\nu}{2}+1\right)}\right)$$

$$\times\left(\frac{G(1+\eta-\frac{\sigma}{2})G(1-\nu)\Big(G(1-\frac{\nu}{2}+\frac{\sigma}{2})\Big)\Big(G(1+\frac{\eta}{2}-3\frac{\sigma}{4})\Big)\Big(G(1-\frac{\eta}{2}-\frac{\sigma}{4})\Big)}{G(1+\frac{\sigma}{2}-\eta)\Big(G(1-\frac{\nu}{2}-\frac{\sigma}{2})\Big)\Big(G(1+\frac{\eta}{2}+\frac{\sigma}{4})\Big)\Big(G(1-\frac{\eta}{2}+3\frac{\sigma}{4})\Big)}\right)^{\frac{1}{2}}$$

We can check, that

$$\chi(\mu+1,\eta) = e^{-i\pi\eta}\chi(\mu,\eta), \quad \text{and} \quad \chi(\nu+2,\rho) = e^{-i\pi\rho}\chi(\nu,\rho)$$

THANK YOU!