Limiting distribution of smallest eigenvalue of thinned complex Wishart matrices

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October, 25th, 2016

Complex Wishart Matrix

- Consider $n \times m$ matrix X, $n \geq m$ with X_{ij} complex valued independent random variables with probability density function

$$\frac{1}{\pi} e^{-|X_{ij}|^2}$$

- One can think about columns of X as independent Gaussian vectors.
- $m \times m$ matrix $X^* X$ is called the uncorrelated complex Wishart matrix.

Eigenvalue p.d.f

- Joint probability density function for eigenvalues of X^*X is

$$p(\lambda_1, \dots, \lambda_m) = \frac{1}{Z_m} \prod_{1 \le k < j \le m} (\lambda_k - \lambda_j)^2 \prod_{\ell=1}^{m} \lambda_\ell^{n-m} e^{-\lambda_\ell}, \quad \lambda_\ell \ge 0.$$

- One can generalize this density for $\alpha > -1$ as

$$p(\lambda_1, \dots, \lambda_m) = \frac{1}{Z_m} \prod_{1 \le k < j \le m} (\lambda_k - \lambda_j)^2 \prod_{\ell=1}^{m} \lambda_\ell^{\alpha} e^{-\lambda_\ell}, \quad \lambda_\ell \ge 0, \tag{1}$$

- The last formula describes Laguerre Unitary Ensemble and can be interpreted as the log gas with Hamiltonian

$$H = - \sum_{1 \le k < j \le m} 2 \ln(\lambda_k - \lambda_j) - \sum_{\ell=1}^{m} (\alpha \ln \lambda_\ell - \lambda_\ell)$$

Generalized Laguerre polynomials

- Generalized Laguerre polynomials are orthogonal polynomials determined by the condition

$$\int_0^{+\infty} L_n^\alpha(x) L_m^\alpha(x) x^\alpha e^{-x} dx = \delta_{n,m} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}.$$

- The probability density function (1) can be rewritten as the determinant of Christoffel Darboux kernel

$$\prod_{k<j} (\lambda_k - \lambda_j)^2 \prod_{\ell=1}^{m} \lambda_\ell^\alpha e^{-\lambda_\ell} = G(m+1) G(m+\alpha+1) \det[K_m(\lambda_j, \lambda_k)]_{j,k=1\ldots m}$$

$$K_m(x,y) = \frac{\Gamma(m+1)(xy)^{\frac{\alpha}{2}} e^{-\frac{x+y}{2}}}{\Gamma(m+\alpha)} \frac{L_{m-1}^\alpha(x) L_m^\alpha(y) - L_{m-1}^\alpha(y) L_m^\alpha(x)}{x-y} \qquad (2)$$

k-point correlation function

- Consider k-point correlation function

$$\rho_k(\mu_1, \dots, \mu_k) = \mathbb{E}\left[\sum_{j_1 \neq \dots \neq j_k} \prod_{\ell=1}^{k} \delta(\mu_\ell - \lambda_{j_\ell})\right]$$

$$= \frac{m!}{(m-k)!} \int_0^{+\infty} \dots \int_0^{+\infty} p(\mu_1, \dots, \mu_k, \lambda_{k+1}, \dots, \lambda_m) d\lambda_{k+1} \dots d\lambda_m$$

- The correlation functions also can be written as determinant. That means that Laguerre Unitary Ensemble is determinantal point process.

$$\rho_k(\mu_1, \dots, \mu_k) = \det[K_m(\mu_j, \mu_\ell)]_{j,\ell=1\dots k} \tag{3}$$

Eigenvalues probabilities

- Let's compute the probability $E_m(\ell,(0,\nu))$ that there are exactly l particles/eigenvalues in the interval $(0,\nu)$.

$$E_m(\ell,(0,\nu)) = \binom{m}{\ell} \times$$

$$\times \int_{\nu}^{+\infty} \dots \int_{\nu}^{+\infty} \left(\int_{0}^{\nu} \dots \int_{0}^{\nu} p(\lambda_1 \dots \lambda_m) d\lambda_1 \dots d\lambda_\ell \right) d\lambda_{\ell+1} \dots d\lambda_m$$

Generating function for eigenvalues probabilities

- Introduce generating function

$$E_m((0,\nu),\xi) = \int_0^{+\infty} \dots \int_0^{+\infty} \prod_{\ell=1}^{m} (1-\xi\chi^{\ell}_{(0,\nu)}) p(\lambda_1 \dots \lambda_m) d\lambda_1 \dots d\lambda_m$$

- It is constructed in such a way, so that

$$E_m(\ell,(0,\nu)) = \frac{(-1)^{\ell}}{\ell!} \frac{\partial^{\ell}}{\partial\xi^{\ell}} E_m((0,\nu),\xi)\Big|_{\xi=1}$$

Generating function is Fredholm determinant

- It also holds, that

$$E_m((0,\nu),\xi) = \sum_{j=0}^{m} \frac{(-\xi)^j}{j!} \int_0^{\nu} \dots \int_0^{\nu} \rho_j(\lambda_1 \dots \lambda_j) d\lambda_1 \dots d\lambda_j$$

- Using the determinant formula (3) we get

$$E_m((0,\nu),\xi) = \det(I - \xi K_m)$$

where K_m is the operator in $L_2((0,\nu))$ with Christoffel-Darboux kernel (2).

Gap probabilities for the thinned ensemble

- Thinning procedure consists of removing every particle with probability $1-\gamma \in (0,1]$. Then the gap probability $E_m^{\gamma}(0,(0,\nu))$ of thinned ensemble is given by

$$E_m^{\gamma}(0,(0,\nu)) = \sum_{j=0}^{m} E_m(j,(0,\nu))(1-\gamma)^j$$

$$= \sum_{j=0}^{m} \frac{(\gamma-1)^j}{j!} \frac{\partial^j}{\partial \xi^j} E_m((0,\nu),\xi)|_{\xi=1} = E_m((0,\nu),\gamma)$$

$$= \det(I-\gamma K_m)$$

Large m limit of Christoffel-Darboux kernel

- We consider large m limit of Laguerre Unitary Ensemble using asymptotics of Generalized Laguerre polynomials in different regions.
- Global limiting density is given by Marchenko-Pastur law

$$\lim_{m\to\infty} \rho_1(m\mu) = \lim_{m\to\infty} K_m(m\mu, m\mu) = \frac{1}{2\pi}\sqrt{\frac{4-\mu}{\mu}}\chi_{(0,4)}(\mu)$$

- We are interested in the smallest eigenvalue. Near the origin the appropriate limit of Christoffel-Darboux kernel is given by

$$\lim_{m\to\infty} \frac{K_m\left(\frac{x}{4m}, \frac{y}{4m}\right)}{4m} = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)} \tag{4}$$

Limiting distribution of smallest eigenvalue of thinned Laguerre Unitary Ensemble

- Taking the large m limit after change of variable in the series expansion of Fredholm determinant gives

$$\lim_{m\to\infty} Pr\left(\lambda^{\gamma}_{\min} \geq \frac{t}{4m}\right) = \lim_{m\to\infty} E^{\gamma}_{m}\left(0, \left(0, \frac{t}{4m}\right)\right) = \det(I - \gamma K_{Bess})$$

where K_{Bess} is the integral operator in $L_2((0,t))$ with the kernel (4).

Asymptotics for small t

- For $\alpha > 0$ using the Fredholm determinant expansion we have

$$\lim_{m\to\infty} Pr\left(\lambda_{\min}^{\gamma} \geq \frac{t}{4m}\right) = \det(I - \gamma K_{Bess})$$

$$= 1 - \gamma \int_0^t K_{Bess}(x,x)dx + O(t^2)$$

$$= 1 - \frac{\gamma}{\Gamma^2(2+\alpha)} \left(\frac{t}{4}\right)^{\alpha+1} (1 + o(1)), \quad t \to 0.$$

Integrable integral operator

- Operator γK_{Bess} is integrable integral operator, since its kernel (4) can be written as

$$\gamma K_{Bess}(x,y) = \frac{\vec{f}^{\,T}(x)\vec{g}(y)}{x-y},$$

where

$$\vec{f}(x) = \begin{pmatrix} \phi(x) \\ \psi(x) \end{pmatrix}, \qquad \vec{g}(x) = \begin{pmatrix} \psi(x) \\ -\phi(x) \end{pmatrix}$$

$$\phi(x) = \sqrt{\frac{\gamma}{2}} J_\alpha(\sqrt{x}), \qquad \psi(x) = \sqrt{\frac{\gamma}{2}} \sqrt{x} J'_\alpha(\sqrt{x}).$$

Resolvent

- For resolvent we have

$$(I - \gamma K_{Bess})^{-1} = I + R,$$

where R is the integral operator with kernel

$$R(x,y) = \frac{\vec{F}^T(x)\vec{G}(y)}{x-y},$$

where

$$F_j = (I - \gamma K_{Bess})^{-1} f_j, \quad G_j = (I - \gamma K_{Bess})^{-1} g_j.$$

Riemann-Hilbert problem

- Consider the matrix-valued function

$$Y(z) = \mathbb{I} - \int_0^t \frac{\vec{F}(x)\vec{g}^T(x)}{x-z}dx$$

- $Y(z)$ is holomorphic function outside of the interval $(0,t)$. On the interval it has boundary values $Y_\pm(z)$.
- Let's notice that for $z \in (0,t)$

$$Y_\pm(z)\vec{f}(z) = \vec{f}(z) - \int_0^t \frac{\vec{F}(x)\vec{g}^T(x)\vec{f}(z)}{x-z\mp i0}dx$$

$$\vec{f}(z) - \int_0^t \frac{\vec{F}(x)\vec{g}^T(x)\vec{f}(z)}{x-z}dx = \vec{f}(z) + (\gamma K_{Bess}\vec{F})(z) = \vec{F}(z).$$

Riemann-Hilbert problem

- Now we have

$$Y_+(z) - Y_-(z) = -2\pi i \vec{F}(z)\vec{g}^T(z) = -2\pi i Y_-(z)\vec{f}(z)\vec{g}^T(z).$$

- That function satisfy **RHP1**
  1. Y(z) is holomorphic outside of the interval $(0, t)$.
  2. $Y_+(z) = Y_-(z)(\mathbb{I} - 2\pi i \vec{f}(z)\vec{g}^T(z))$
  3. $Y(z)$ is integrable near end points of the interval.
  4. $Y(z) = \mathbb{I} + O(z^{-1}), \quad z \to \infty$
- Solution to such Riemann-Hilbert problem is unique.

Relation to Fredholm Determinant

- We can rescale the variables to get

$$\det(\mathbb{I} - \gamma K_{Bess}) = \det(\mathbb{I} - \gamma \tilde{K}_{Bess})$$

where $\tilde{K}_{Bess}$ acts on $L_2((0,1))$ with the kernel

$$\tilde{K}_{Bess}(x,y) = \frac{J_\alpha(\sqrt{tx})\sqrt{ty}J'_\alpha(\sqrt{ty}) - \sqrt{tx}J'_\alpha(\sqrt{tx})J_\alpha(\sqrt{ty})}{2(x-y)}$$

- Using the Bessel equation we have

$$\frac{d}{dt}\tilde{K}_{Bess}(x,y) = \frac{1}{4}J_\alpha(\sqrt{tx})J_\alpha(\sqrt{ty}).$$

Relation to Fredholm Determinant

- We get

$$\frac{d}{dt}\ln\det(\mathbb{I}-\gamma K_{Bess}) = -\gamma\, Tr\left((\mathbb{I}-\gamma \tilde{K}_{Bess})^{-1}\frac{d}{dt}\tilde{K}_{Bess}\right)$$

$$= \frac{1}{2}\int_0^t F_1(x)g_2(x)dx = \frac{(Y_1)_{1,2}}{2},$$

where

$$Y(z) = \mathbb{I} + \frac{Y_1}{z} + O(z^{-2}), \quad z \to \infty$$

Deformation of Riemann-Hilbert problem

- We can notice that from Bessel equation it follows

$$\frac{d\vec{f}}{dx} = \begin{pmatrix} 0 & \dfrac{1}{2x} \\ \dfrac{\alpha^2}{2x} - \dfrac{1}{2} & 0 \end{pmatrix} \vec{f}.$$

- Since the coefficient matrix has zero trace, we can find the fundamental solution with determinant 1, such that its first column is proportional to $\vec{f}$. Take
$\Psi_\alpha(z) =$
$$\sqrt{\pi} e^{-i\frac{\pi}{4}} \begin{pmatrix} \mathrm{I}_\alpha(\sqrt{e^{-i\pi} z}) & -\frac{i}{\pi}\mathrm{K}_\alpha(\sqrt{e^{-i\pi} z}) \\ \sqrt{e^{-i\pi} z}\, \mathrm{I}'_\alpha(\sqrt{e^{-i\pi} z}) & -\frac{i}{\pi}\sqrt{e^{-i\pi} z}\mathrm{K}'_\alpha(\sqrt{e^{-i\pi} z}) \end{pmatrix}$$

Deformation of Riemann-Hilbert problem

$\Psi_\alpha(z)$ satisfies the **RHP2**, solution to which is unique

1. $\Psi_\alpha(z)$ is holomorphic outside $[0,+\infty)$
2. $\Psi_{\alpha,+}(z)=\Psi_{\alpha,-}(z)\begin{pmatrix} e^{-i\pi\alpha} & 1 \\ 0 & e^{i\pi\alpha}\end{pmatrix}$
3. $$\Psi_\alpha(z)=\begin{pmatrix} 1 & 0 \\ -\frac{4\alpha^2+3}{8} & 1\end{pmatrix}\left(I+O\left(z^{-1}\right)\right)\left(e^{-i\pi}z\right)^{-\frac{1}{4}\sigma_3}\times$$ $$\times\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1\end{pmatrix}e^{-i\frac{\pi}{4}\sigma_3}\exp\left(\left(e^{-i\pi}z\right)^{\frac{1}{2}}\sigma_3\right),\quad z\to\infty,$$
4. $$\Psi_\alpha(z)=\hat{\Psi}_\alpha(z)\left(e^{-i\pi}z\right)^{\frac{\alpha}{2}\sigma_3}\begin{pmatrix} 1 & -\frac{i}{2}\frac{1}{\sin(\pi\alpha)} \\ 0 & 1\end{pmatrix},$$ $z\to 0$, $\alpha\notin\mathbb{Z}$, $$\Psi_\alpha(z)=\hat{\Psi}_\alpha(z)\left(e^{-i\pi}z\right)^{\frac{\alpha}{2}\sigma_3}\left[I-\frac{e^{i\pi\alpha}}{2\pi i}\ln(e^{-i\pi}z)\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\right],$$ $z\to 0$, $\alpha\in\mathbb{Z}$,

Deformation of Riemann-Hilbert problem

- Consider

$$X(z) = \begin{pmatrix} 1 & 0 \\ \frac{4\alpha^2+3}{8} & 1 \end{pmatrix} Y(zt)\Psi_\alpha(zt)$$

- Then $X(z)$ satisfies **RHP3**
  1. $X(z)$ is holomorphic outside $[0, +\infty)$
  2. $X_+(z) = X_-(z) \begin{pmatrix} e^{-i\pi\alpha} & 1 \\ 0 & e^{i\pi\alpha} \end{pmatrix}, \quad z > 1$

     $X_+(z) = X_-(z) \begin{pmatrix} e^{-i\pi\alpha} & 1-\gamma \\ 0 & e^{i\pi\alpha} \end{pmatrix}, \quad 0 < z < 1$
  3. $X(z) = \left(I + O\left(z^{-1}\right)\right)\left(e^{-i\pi} zt\right)^{-\frac{1}{4}\sigma_3} \times$

     $\times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} e^{-i\frac{\pi}{4}\sigma_3} \exp\left(\left(e^{-i\pi} zt\right)^{\frac{1}{2}} \sigma_3\right), \quad z \to \infty,$

Deformation of Riemann-Hilbert problem

4. $X(z) = \hat{X}_0(z)\left(e^{-i\pi}z\right)^{\frac{\alpha}{2}\sigma_3}\begin{pmatrix} 1 & -\frac{i}{2}\frac{1-\gamma}{\sin(\pi a)} \\ 0 & 1 \end{pmatrix}, \ z \to 0, \ \alpha \notin \mathbb{Z},$

   $X(z) = \hat{X}_0(z)\left(e^{-i\pi}z\right)^{\frac{\alpha}{2}\sigma_3} \times$
   $\left[I - \frac{e^{i\pi\alpha}}{2\pi i}(1-\gamma)\ln(e^{-i\pi}z)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right], \ z \to 0, \ \alpha \in \mathbb{Z},$

5. $X(z) = \hat{X}_1(z)\left(I + \frac{\gamma}{2\pi i}\begin{pmatrix} -1 & -e^{-i\pi\alpha} \\ e^{i\pi\alpha} & 1 \end{pmatrix}\ln(z-1)\right) \times$

   $\times \begin{cases} \begin{pmatrix} e^{-i\pi\alpha} & 1 \\ -1 & 0 \end{pmatrix}, & \Im z > 0 \\ \begin{pmatrix} 1 & 0 \\ -e^{i\pi\alpha} & 1 \end{pmatrix}, & \Im z < 0 \end{cases}, \ z \to 1.$

Result of asymptotic analysis

- After that we need to open lenses and construct parametrices to find asymptotics $t \to \infty$. We find asymptotics of $(Y_1)_{1,2}$ and hence of the Fredholm determinant.

$$\lim_{m\to\infty} Pr\left(\lambda_{\min}^{\gamma} \geq \frac{t}{4m}\right) = \det(I - \gamma K_{Bess})$$

$$= C(\alpha,\gamma) e^{-\frac{v}{\pi}\sqrt{t}} t^{\frac{v^2}{8\pi^2}} (1 + o(1))$$

where $v = -\ln(1-\gamma)$.

Isomonodromic deformation equations

- The jump matrix for $X(z)$ is independent from z and t. Therefore its logarithmic derivative is rational function of z.

$$\frac{dX}{dz} = \left(A + \frac{B}{z} + \frac{C}{z-1} \right) X, \quad \frac{dX}{dt} = \left(E + \frac{F}{z} \right) X. \qquad (5)$$

- Matrices A,B,C,E,F can be expressed in terms of parameters: q,p,α and t where

$$q^2 = t(Y_1)_{12}^2 + 2((Y_1)_{11} - (Y_1)_{22}),$$
$$p^2 = \frac{\alpha^2 q^2}{(q^2-1)^2} + \frac{2t}{q^2-1}\left((Y_1)_{12} + \frac{q^2}{2} \right)$$

- Compatibility condition of (5) is called isomonodromic deformation equation.

Hamiltonian equations and action integral

- Isomonodromic deformation equation and is Hamiltonian with

$$H(p,q,t,\alpha) = \frac{(Y_1)_{12}}{2} = \frac{q^2-1}{4t}p^2 - \frac{q^2}{4} - \frac{\alpha^2}{4t(q^2-1)}$$

- We can rewrite representation for the determinant as

$$\ln\left(\det\left(I-\gamma K_{Bess}\right)\right) = \int_0^t H ds = \int_0^t (pq' - H) ds + 2Ht + L$$

Where

$$L = -\frac{\alpha}{2}\ln\left(-(\hat{X}_0(0))_{11}(\hat{X}_0(0))_{12}^{-1}s^{-\alpha}\right)\Big|_{s=0}^{t}$$

Another determinant representation

- We can rewrite the action integral

$$\ln\left(\det\left(I-\gamma K_{\mathrm{Bess}}\right)\right)=\int_{0}^{\gamma}\left(p\frac{\partial q}{\partial\gamma}\right)d\tilde{\gamma}+2Ht+L.$$

- Now we don't have integral with respect to t and we can plug in computed asymptotics.

Final result

$$\lim_{m\to\infty} \mathbb{P}\left(\lambda_{\min}(\gamma) \geq \frac{t}{4m}\right) = e^{-\frac{v}{\pi}\sqrt{t}}(16t)^{\frac{v^2}{8\pi^2}} e^{\frac{\alpha v}{2}} \times$$

$$\times \left| G\left(1 + \frac{iv}{2\pi}\right)\right|^2 (1 + o(1)), \quad t \to +\infty,$$

where $v = -\ln(1-\gamma)$ and $G(\lambda)$ is the Barnes G-function.

THANK YOU